The Resonance Method: Evaluation of Self-Sustainment Voltage

Ing. Filip Zmeko, doc. Ing. Eva Müllerová, Ph.D., Ing. Petr Martínek, Ph.D.

University of West Bohemia in Pilsen (UWB)
Faculty of Electrical Engineering (FEE)

FAKULTA ELEKTROTECHNICKÁ
ZÁPADOČESKÉ UNIVERZITY
V PLZNI

High-voltage laboratory

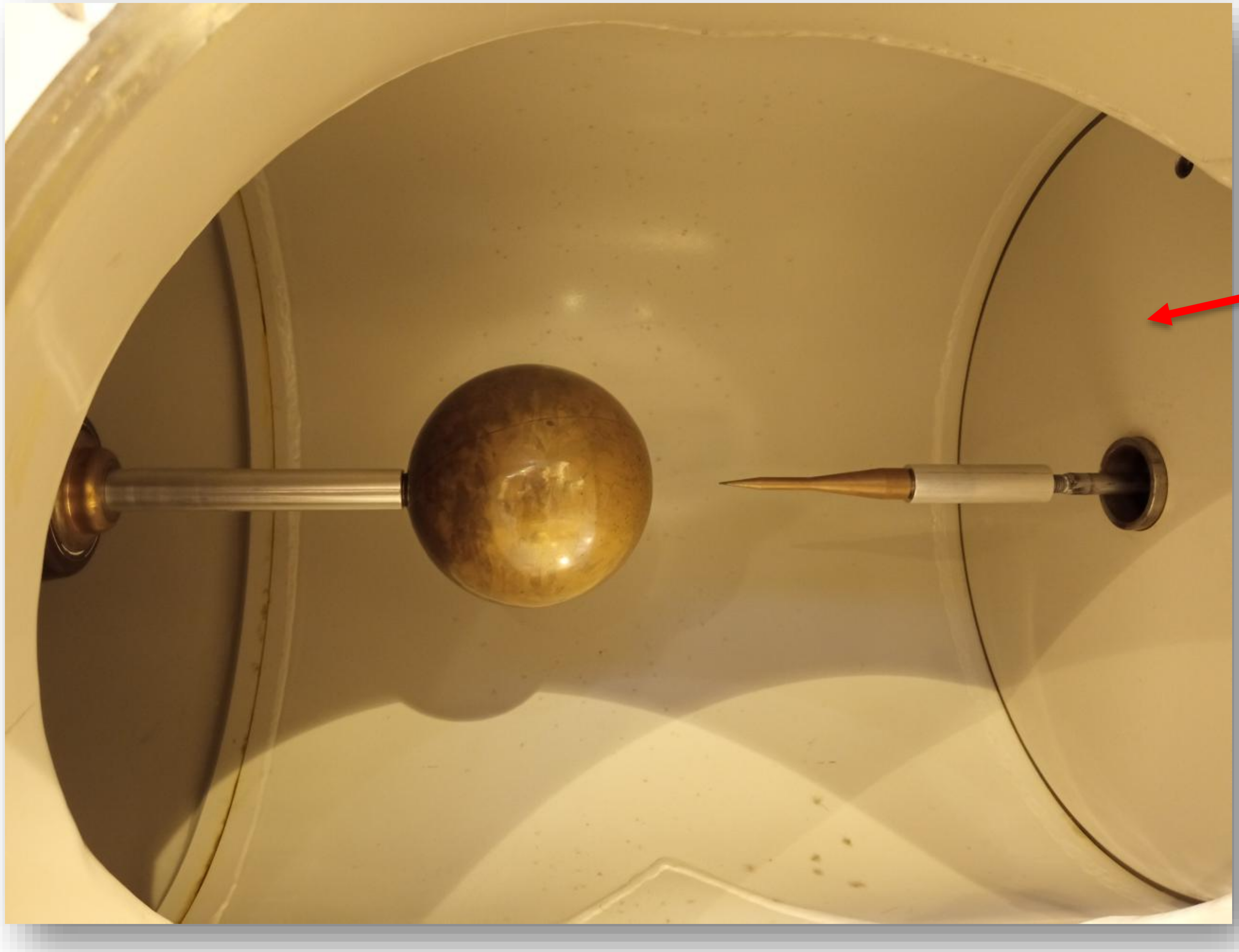


Fig. 1: Pressure vessel interior and electrodes.

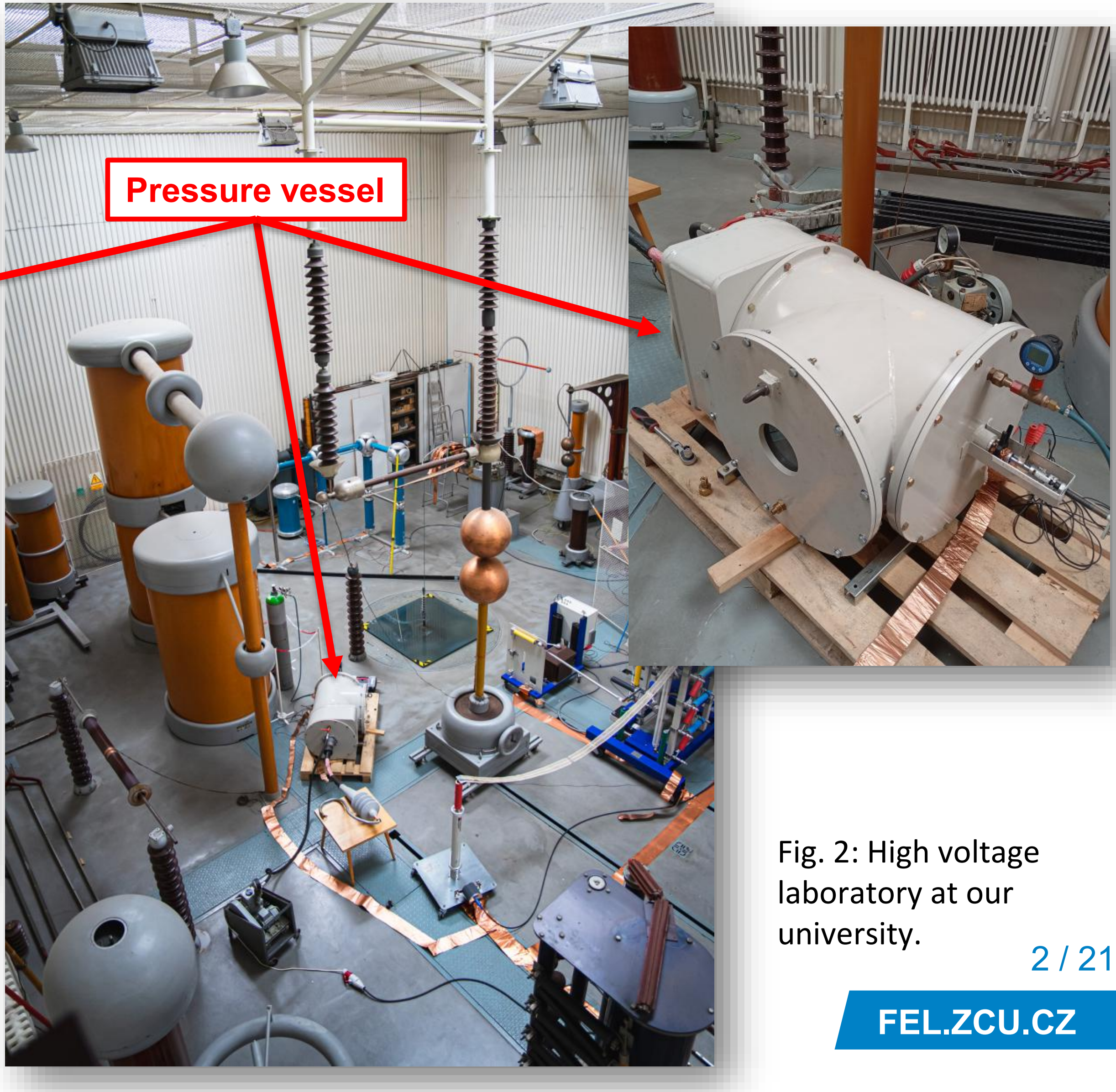


Fig. 2: High voltage laboratory at our university.

High-voltage laboratory

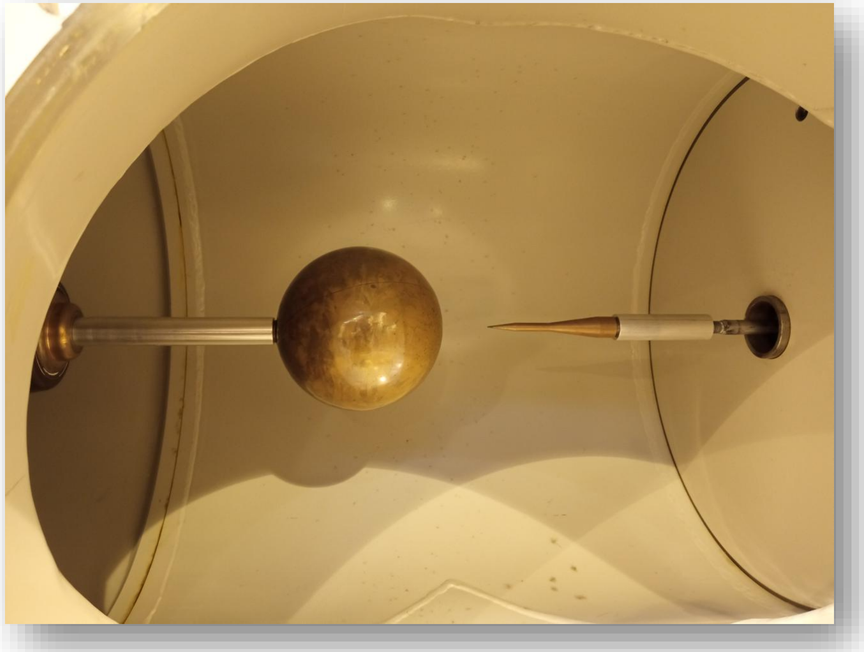


Fig. 1: Pressure vessel interior and electrodes.

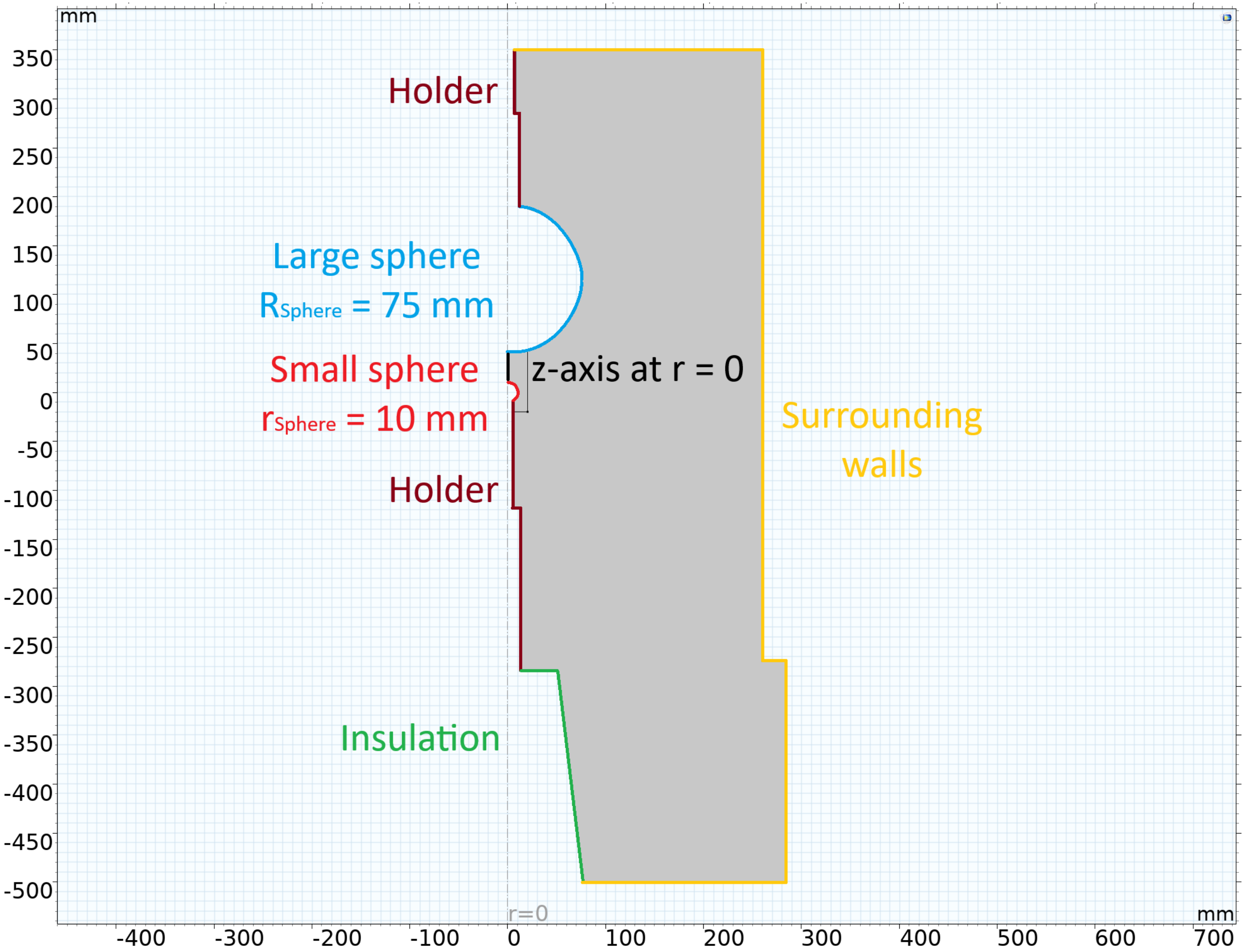


Fig. 3: Axisymmetric geometry in COMSOL Multiphysics, created using [1].

Mathematical model – Charged particle transport

- **Drift-diffusion-reaction equations** (*Transport of Diluted Species*):

$$\frac{\partial n_c}{\partial t} + \nabla \cdot \left[-D_c\left(E/_N\right)\nabla n_c \mp \mu_c\left(E/_N\right) n_c \nabla\varphi \right] = S_c$$

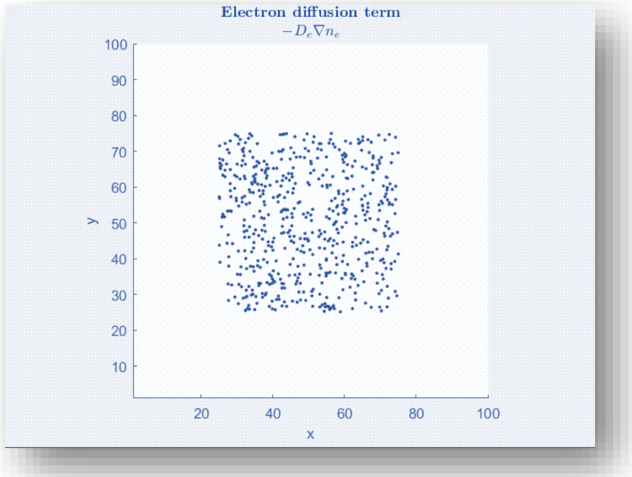


Fig. 4: Particle diffusion, created using [2].

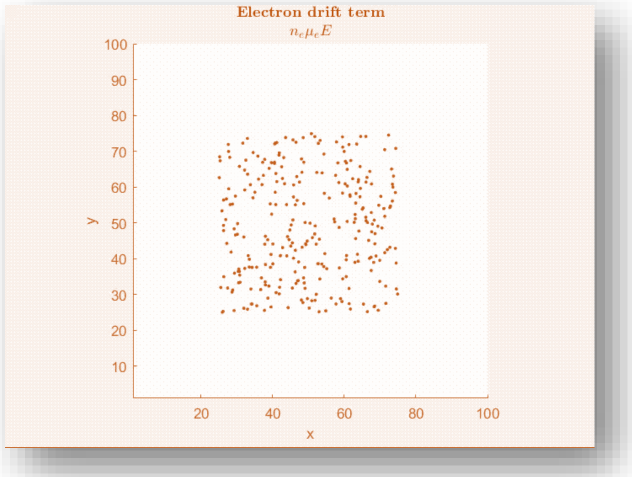


Fig. 5: Particle drift in electric field, created using [2].

Tab. 1: Variable definitions used in the drift-diffusion-reaction-equations.

n_c	Concentrations of particle species c $(\mathrm{m^{-3}})$ – electrons and positive ions
D_c	Diffusivities of particle species c $(\mathrm{m^2 s^{-1}})$
μ_c	Mobilities of particle species c – drift $(\mathrm{m^2 V^{-1} s^{-1}})$
N	Neutral gas particle concentration $(\mathrm{m^{-3}})$
E	Electric field intensity $(\mathrm{Vm^{-1}})$ – $E/_N$ is reduced el. field (Td)
S_c	Source term of particle species c – reactions $(\mathrm{m^{-3} s^{-1}})$

Mathematical model – Charged particle transport

- **Drift-diffusion-reaction equation for electrons:**

$$\nabla\cdot\left[-D_{\mathrm{e}}\left({}^{E}/_{N}\right)\nabla n_{\mathrm{e}}+\mu_{\mathrm{e}}\left({}^{E}/_{N}\right)n_{\mathrm{e}}\nabla\varphi\right]=S_{\mathrm{e}}$$

- Boundary conditions are set as follows:
  - *Flux* with convection term for upper larger sphere ($R_{\mathrm{Sphere}} = 75\ \mathrm{mm}$), holder and surrounding walls

See slide 8

$$J_{0,n_{\mathrm{e}}}=\gamma\mu_{\mathrm{e}}U(es.Er\cdot nr\ +\ es.Ez\cdot nz)n_{\mathrm{p}}-\frac{n_{\mathrm{e}}}{2}\sqrt{\frac{8k_{\mathrm{B}}T_{\mathrm{e}}}{\pi m_{\mathrm{e}}}}$$

  - *Outflow* for lower smaller sphere ($r_{\mathrm{Sphere}} = 10\ \mathrm{mm}$) and holder
  - *Axial symmetry at* $r = 0\ \mathrm{mm}$
  - *No flux* on the insulation

Tab. 2: Variable definitions used for boundary conditions.

γ	Secondary electron emission coefficient (0.0001)
T_{e}	Electron temperature (K)
m_{e}	Electron mass (kg)
k_{B}	Boltzmann constant ($\mathrm{JK^{-1}}$)
nr, nz	Normal components
$es.Er(z)$	Electric field in $r(z)$ direction ($\mathrm{Vm^{-1}}$)

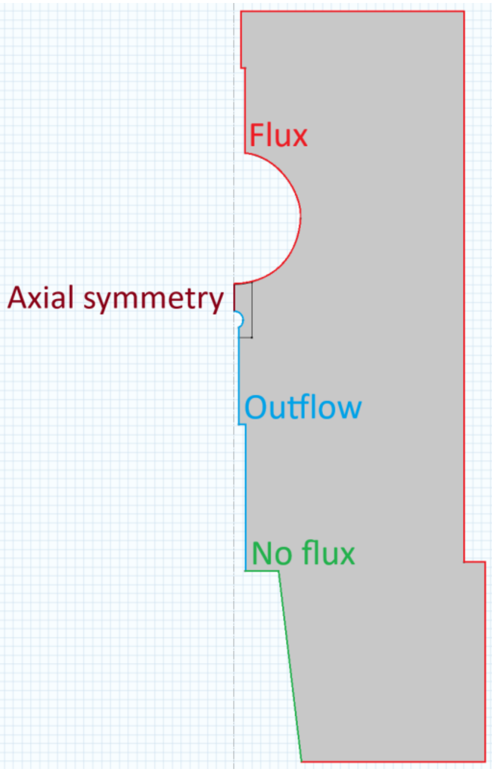


Fig. 6: Domain and boundary conditions, created using [1].

Mathematical model – Charged particle transport

- **Drift-diffusion-reaction equation for positive ions:**

$$\nabla \cdot \left[-D_\mathrm{p}\nabla n_\mathrm{p} - \mu_p n_\mathrm{p}\nabla\varphi\right] = S_\mathrm{p}$$

- Boundary conditions are set as follows:
  - *Flux* with convection term for lower smaller sphere ($r_\mathrm{Sphere} = 10\ \mathrm{mm}$) and holder

$$J_{0,n_\mathrm{p}} = -\frac{n_\mathrm{p}}{2}\sqrt{\frac{8k_\mathrm{B}T_\mathrm{p}}{\pi m_\mathrm{p}}}$$

  - *Outflow* for upper larger sphere ($R_\mathrm{Sphere} = 75\ \mathrm{mm}$), holder and surrounding walls
  - *Axial symmetry* at $r = 0\ \mathrm{mm}$
  - *No flux* on the insulation

Tab. 3: Variable definitions used for boundary conditions.

T_p	Positive ion temperature (K) – set equal to $300\ \mathrm{K}$
m_p	Positive ion mass (kg)

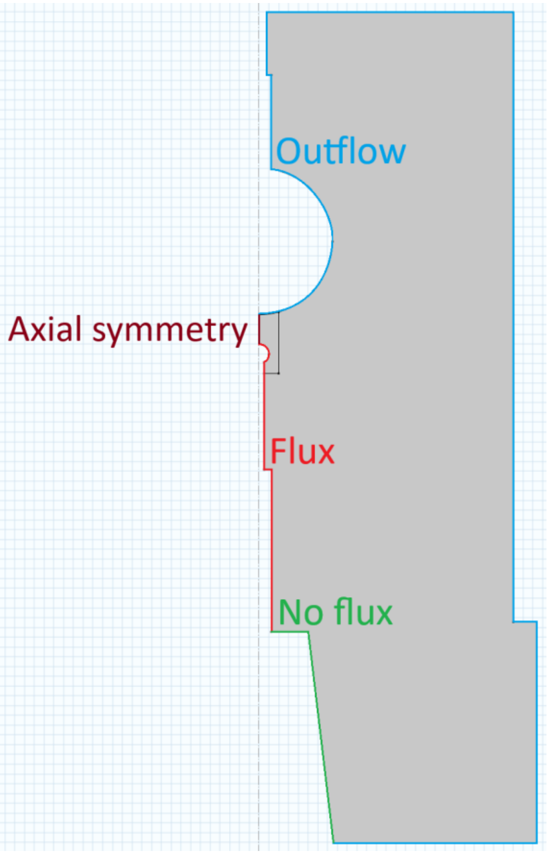


Fig. 7: Domain and boundary conditions, created using [1].

Mathematical model – Charged particle transport

- Source term for electrons:

$$S_{\mathrm{e}} = W + S_{\mathrm{coll(eff)}} + S_{\mathrm{ph}}$$

$$S_{\mathrm{coll(eff)}} = \alpha_{\mathrm{eff}}\left({}^{E}/_{N}\right) \cdot N \cdot \mu_{\mathrm{e}}\left({}^{E}/_{N}\right) \cdot E \cdot n_{\mathrm{e}}$$

- Source term for positive ions:

$$S_{\mathrm{p}} = W + S_{\mathrm{coll}} + S_{\mathrm{ph}}$$

$$S_{\mathrm{coll}} = \alpha\left({}^{E}/_{N}\right) \cdot N \cdot \mu_{\mathrm{e}}\left({}^{E}/_{N}\right) \cdot E \cdot n_{\mathrm{e}}$$

- W is an independent source term set equal to $100\ \mathrm{m}^{-3}\mathrm{s}^{-1}$
- S_{ph} is photoionization rate, see slide 9
- α and α_{eff} are ionization coefficients defined according to [3]
- All transport coefficients for electrons and electron temperature are defined according to [4]

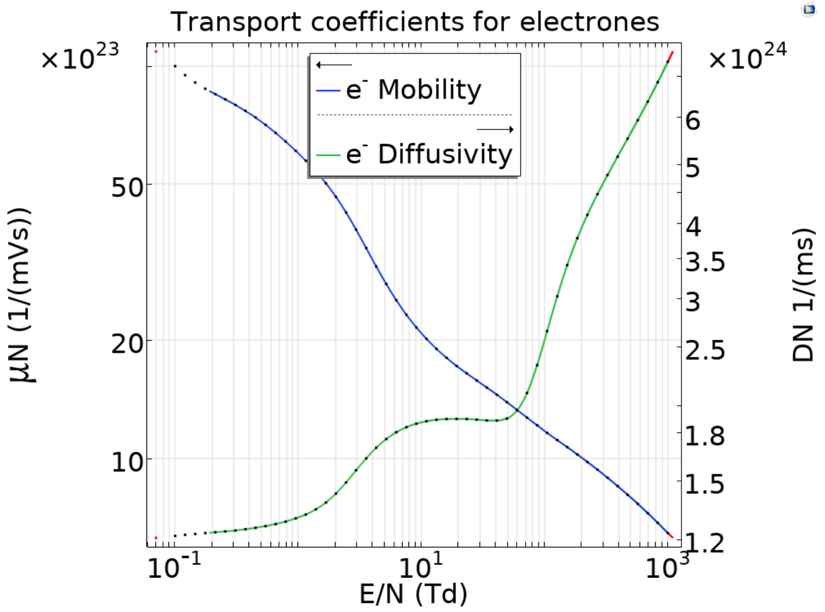


Fig. 8: Transport (electron e mobility μ_{e} and diffusivity D_{e}) coefficients as a functions of reduced electric field ${}^{E}/_{N}$ (Td) [4].

Tab. 4: Transport coefficients for positive ions p [5].

μ_{p}	2.0e-4 [m^2/V/s]	Positive ion p mobility
D_{p}	5.05e-6 [m^2/s]	Positive ion p diffusivity

Mathematical model – Electric field

- **Laplace equation for electrostatic potential** (*Electrostatics*): $\Delta\varphi = 0$
- **Electrostatic field intensity**: $\boldsymbol{E} = -\nabla\varphi$
- **Space charge** does not disturb external electric field – linear problem
- **External electric field** is calculated only once and then scaled by voltage U
- Boundary conditions are set as follows:
  - 1 V for lower smaller sphere ($r_{\mathrm{Sphere}} = 10\ \mathrm{mm}$) and holder
  - *Ground* for upper larger sphere ($R_{\mathrm{Sphere}} = 75\ \mathrm{mm}$), holder and surrounding walls
  - *Axial symmetry at* $r = 0\ \mathrm{mm}$
  - *Zero charge* on the insulation

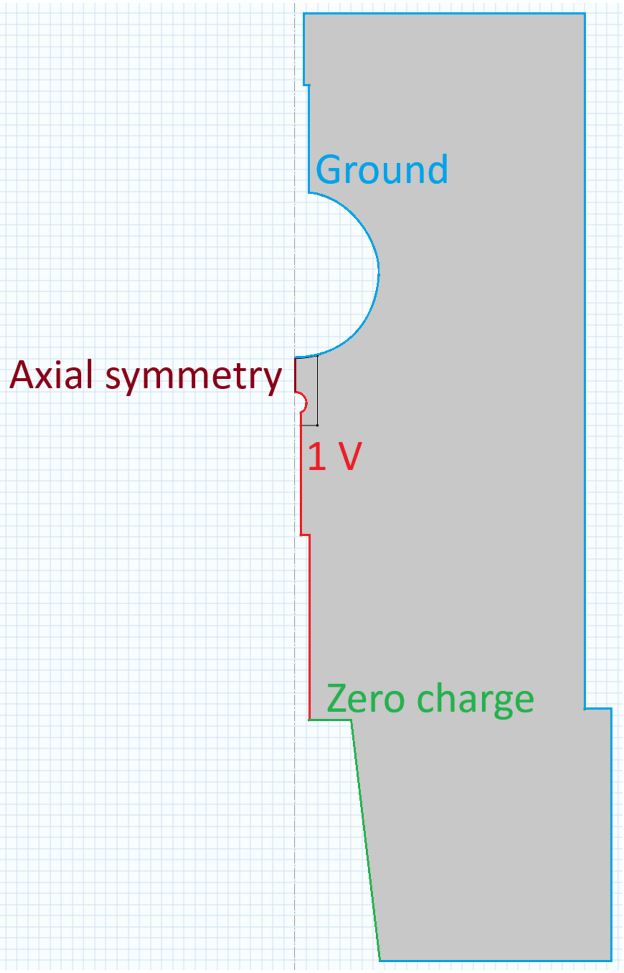


Fig. 9: Domain and boundary conditions, created using [1].

Mathematical model – Photoionization

- **Three-exponential Helmholtz** model for photoionization rate $S_{\mathrm{ph},i}$ proposed by A. Bourdon, et al. in [6] (*Coefficient Form PDE*):

$$\Delta S_{\mathrm{ph},i} - \left(\lambda_i \cdot p_{\mathrm{O_2}}\right)^2 \cdot S_{\mathrm{ph},i} = -A_i \cdot p_{\mathrm{O_2}}^2 \cdot \chi_{\mathrm{ext}}\left({}^{E}\!/_{N}\right) \cdot f_{\mathrm{q}} \cdot S_{\mathrm{coll}}$$

$$S_{\mathrm{ph}} = \sum_{i=1}^{3} S_{\mathrm{ph},i}$$

- Quenching factor f_{q}, partial pressure of molecular oxygen $p_{\mathrm{O_2}}$ and parameters λ_i and A_i (where $i = 1, 2, 3$) are summarised in the Table 5 and collisional ionization S_{coll} is defined as follows:

$$S_{\mathrm{coll}} = \alpha\left({}^{E}\!/_{N}\right) \cdot N \cdot \mu_{\mathrm{e}}\left({}^{E}\!/_{N}\right) \cdot E \cdot n_{\mathrm{e}}$$

$${}^{E}\!/_{N} = \left(1e21 \cdot U \cdot \frac{es.normE}{N}\right) \text{ and } E = U \cdot es.normE$$

- Boundary conditions are set according to Fig. 10

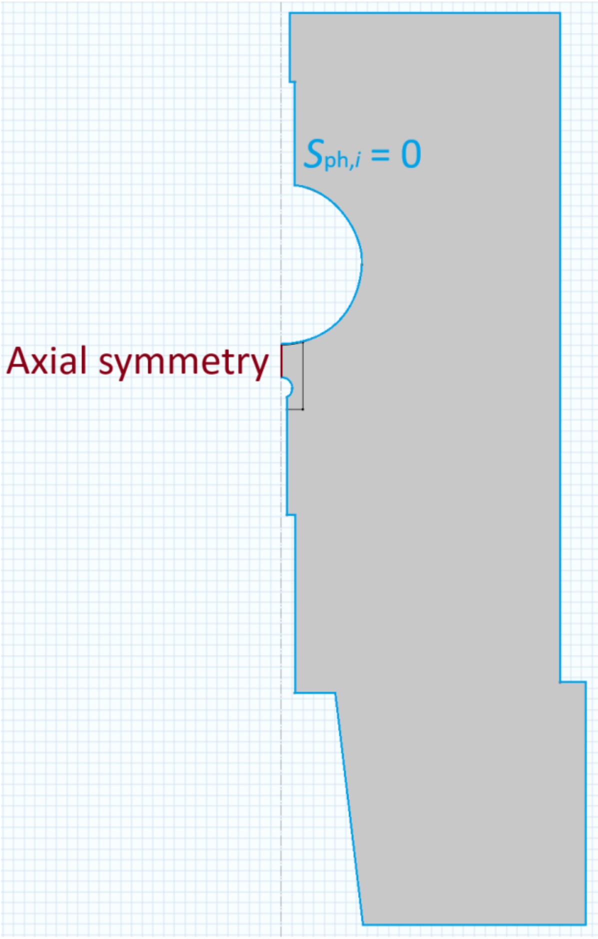


Fig. 10: Domain and boundary conditions, created using [1].

Tab. 5: Photoionization model parameters [6].

A_1	1.986e-4 [cm^-2*Torr^-2]
A_2	0.0051 [cm^-2*Torr^-2]
A_3	0.4886 [cm^-2*Torr^-2]
λ_1	0.0553 [cm^-1*Torr^-1]
λ_2	0.1460 [cm^-1*Torr^-1]
λ_3	0.89 [cm^-1*Torr^-1]
$p_{\mathrm{O_2}}$	150 [Torr]
f_{q}	0.037975

Mathematical model – Photoionization

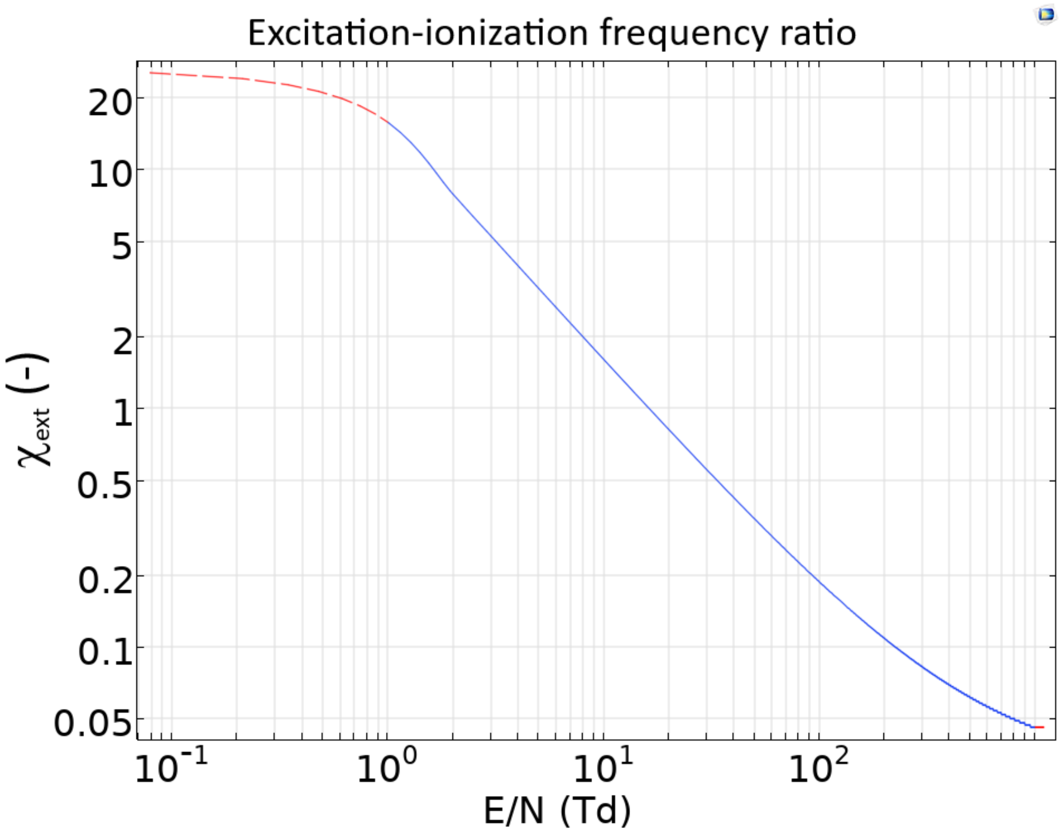


Fig. 11: Excitation-ionization frequency ratio χ_{ext} from slide 9 as a functions of reduced electric field $^{E}/_{N}$ (Td) for the photoionization model [6].

The Resonance Method

- Method for calculating self-sustainment voltage U_0 – transition between non-self-sustaining and self-sustaining discharge – proposed by prof. M. S. Benilov, et al. in [7]
- **Linear stationary ($\partial/\partial t \rightarrow 0$) study** for the resonance of current I
  - Parametric sweep for voltage U
  - Once the current I diverges, the resonance is found at U_0

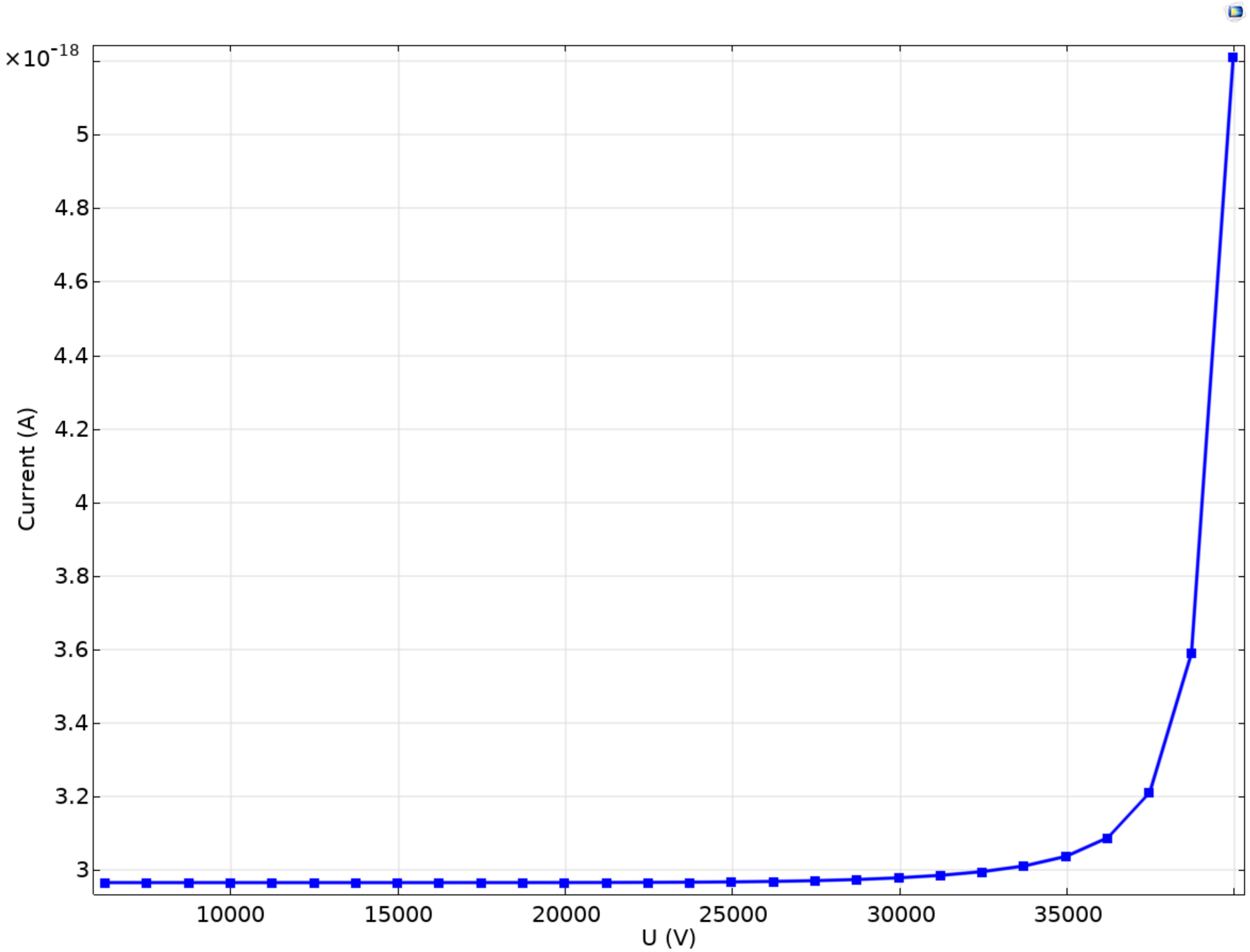


Fig. 12: Current-voltage characteristic of non-self-sustained discharge for air, $d = 30\ \mathrm{mm}$ and $p = 1\ \mathrm{atm}$, created using [1].

The Resonance Method

- Method for calculating self-sustainment voltage U_0 – transition between non-self-sustaining and self-sustaining discharge – proposed by prof. M. S. Benilov, et al. in [7]
- **Linear stationary ($\partial/\partial t \rightarrow 0$) study** for the resonance of current I
  - Parametric sweep for voltage U
  - Once the current I diverges, the resonance is found at U_0
  - Particle concentrations switches to negative (non-physical) values for $U > U_0$ to ensure the solution continues for any value of U
  - To model discharge behaviour for $U > U_0$ nonlinear coupling must be added

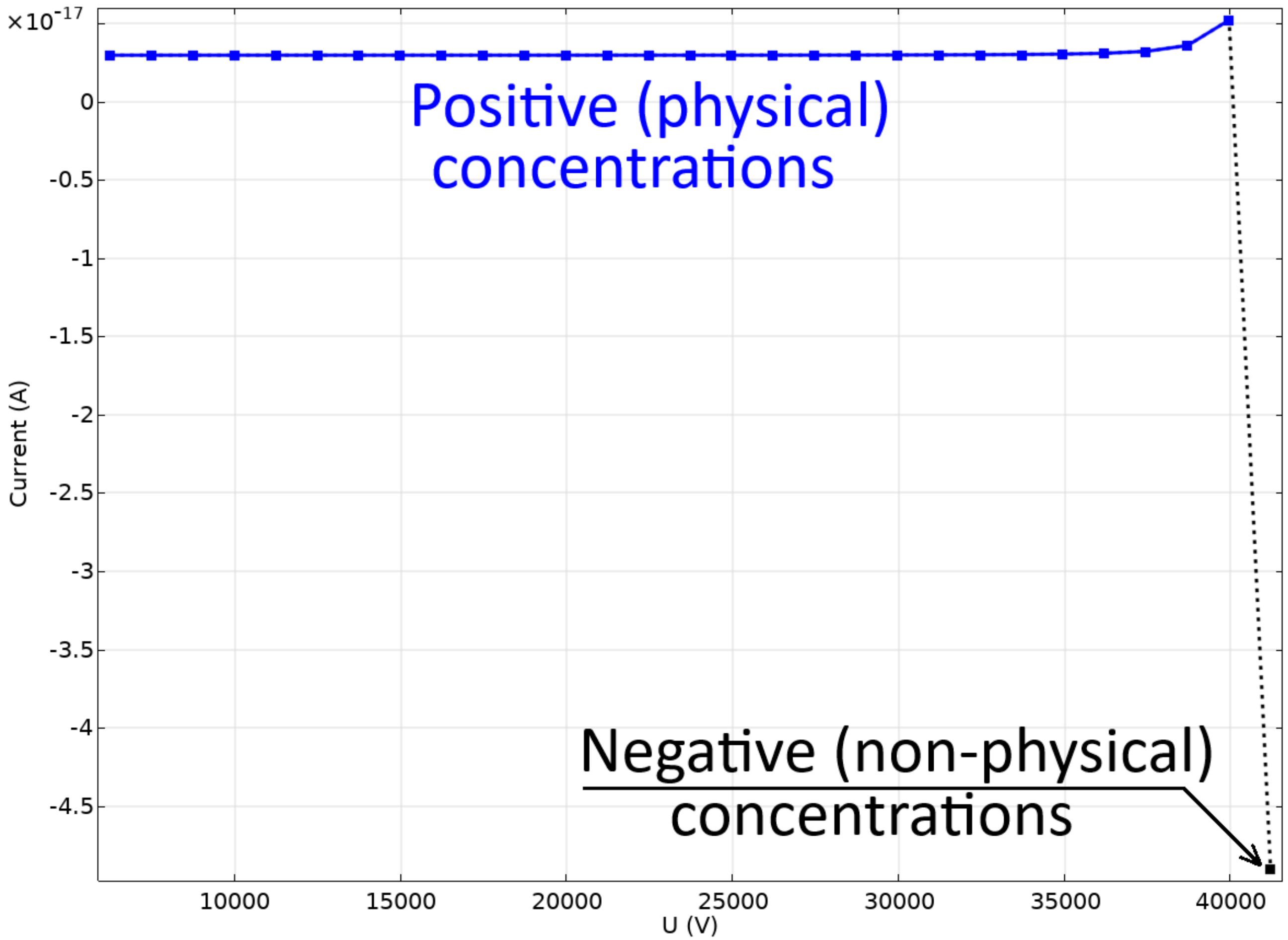
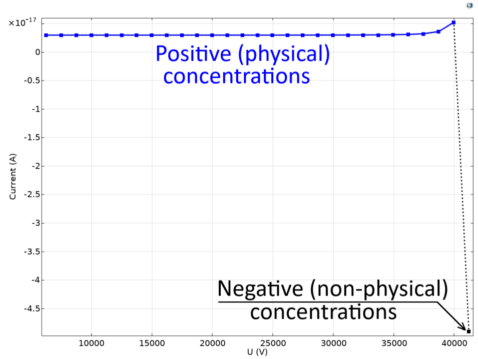


Fig. 13: Current-voltage characteristic of non-self-sustained discharge with resonance point for air, $d = 30$ mm and $p = 1$ atm, created using [1].

The Resonance Method

- Method for calculating self-sustainment voltage U_0 – transition between non-self-sustaining and self-sustaining discharge – proposed by prof. M. S. Benilov, et al. in [7]
- **Linear stationary ($\partial/\partial t \rightarrow 0$) study** for the resonance of current I
  - Parametric sweep for voltage U
  - Once the current I diverges, the resonance is found at U_0



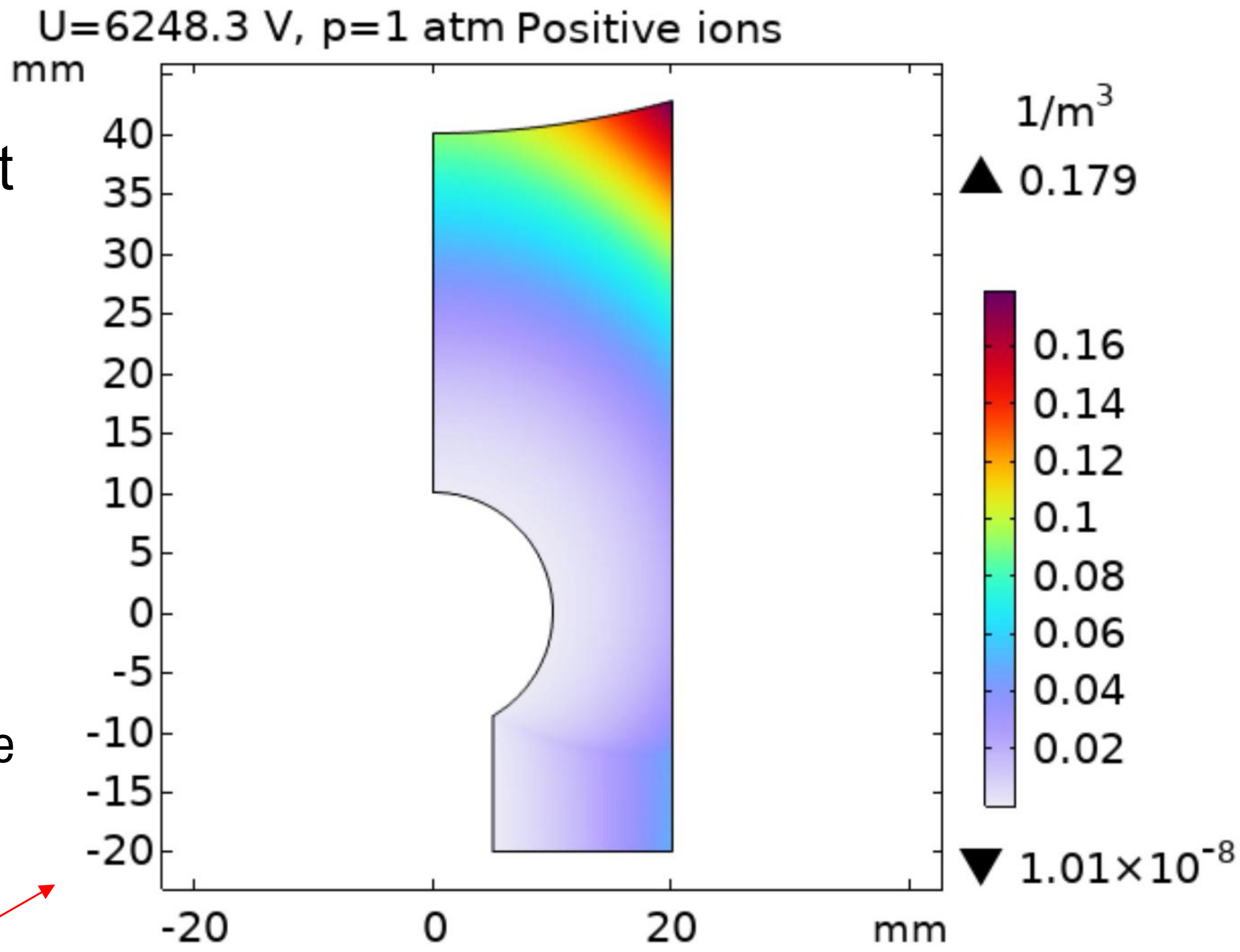


Fig. 14: Concentration distribution of positive ions for voltage values in Fig. 13 for air, $d = 30$ mm and $p = 1$ atm (animation), created using [1].

The Resonance Method

- Method for calculating self-sustainment voltage U_0 – transition between non-self-sustaining and self-sustaining discharge – proposed by prof. M. S. Benilov, et al. in [7]
- **Linear stationary ($\partial/\partial t \rightarrow 0$) study** for the resonance of current I
  - Parametric sweep for voltage U
  - Once the current I diverges, the resonance is found at U_0
  - Particle concentrations switches to negative (non-physical) values for $U > U_0$ to ensure the solution continues for any value of U
  - To model discharge behaviour for $U > U_0$ nonlinear coupling must be added

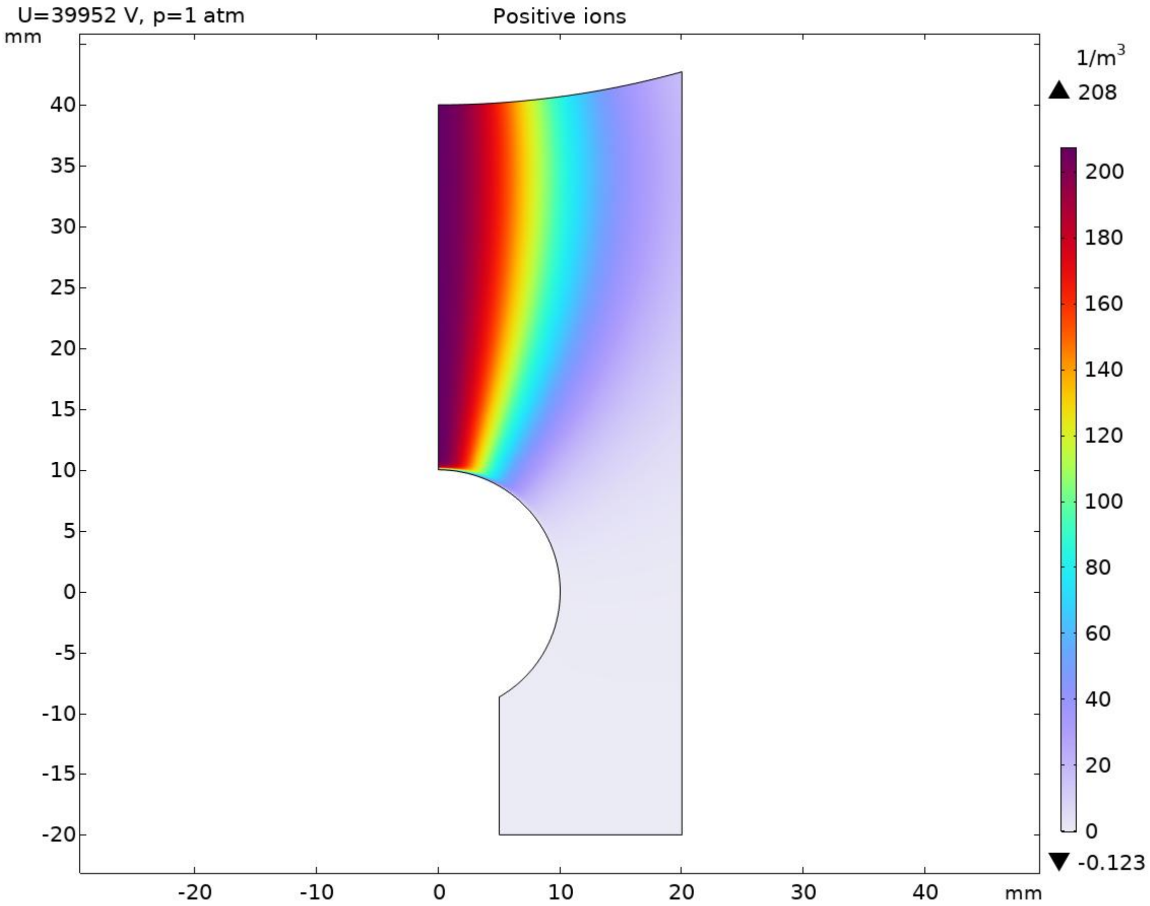


Fig. 15: Concentration distribution of positive ions for voltage just below the resonance point for air, $d = 30\ \mathrm{mm}$ and $p = 1\ \mathrm{atm}$ – positive (physical) values, created using [1].

The Resonance Method

- Method for calculating self-sustainment voltage U_0 – transition between non-self-sustaining and self-sustaining discharge – proposed by prof. M. S. Benilov, et al. in [7]
- **Linear stationary ($\partial/\partial t \rightarrow 0$) study** for the resonance of current I
  - Parametric sweep for voltage U
  - Once the current I diverges, the resonance is found at U_0
  - Particle concentrations switches to negative (non-physical) values for $U > U_0$ to ensure the solution continues for any value of U
  - To model discharge behaviour for $U > U_0$ nonlinear coupling must be added

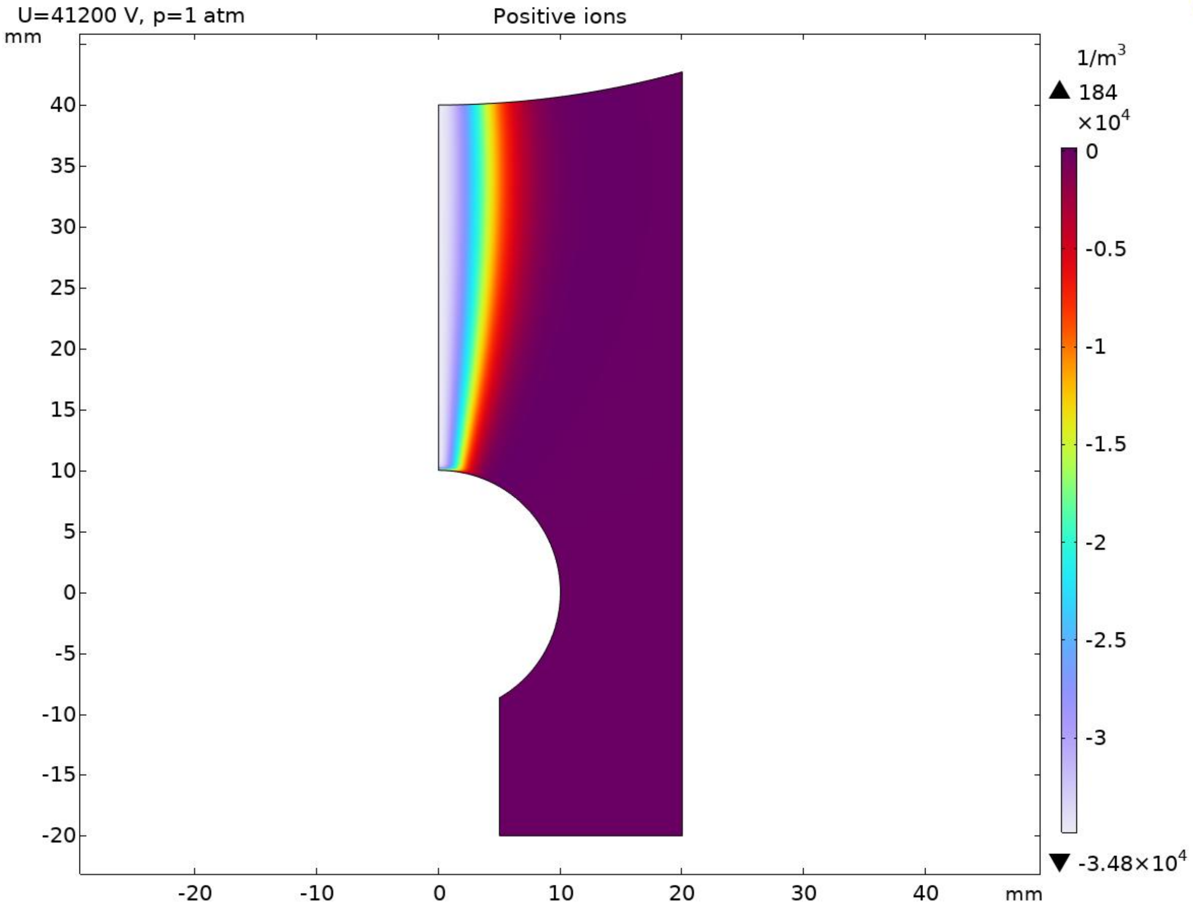


Fig. 16: Concentration distribution of positive ions for voltage at the resonance point for air, $d = 30\ \mathrm{mm}$ and $p = 1\ \mathrm{atm}$ – negative (non-physical) values, created using [1].

The Resonance Method

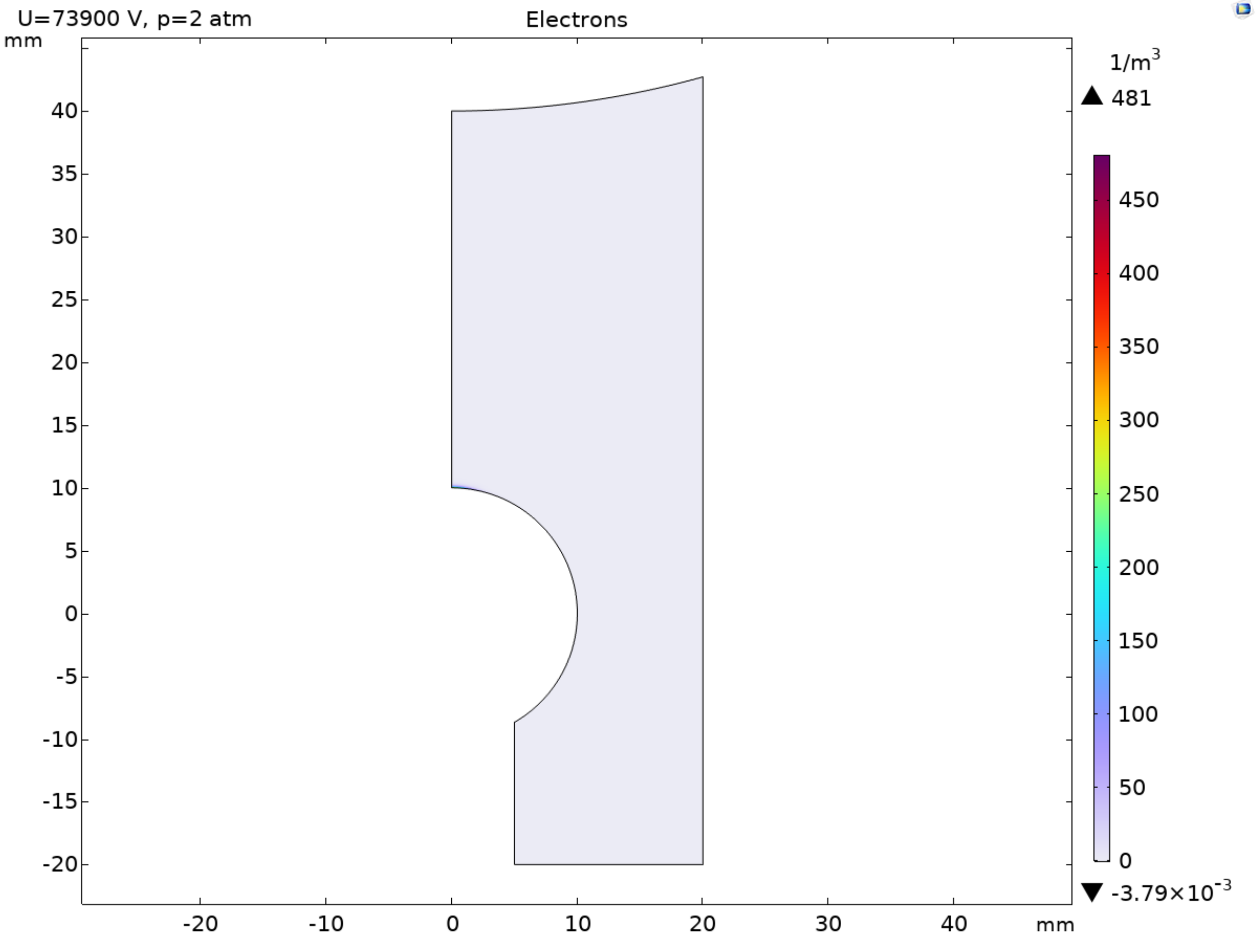


Fig. 17: Concentration distribution of electrons for voltage just below the resonance point for air, $d = 30\ \mathrm{mm}$ – positive (physical) values, created using [1]. (Note that $p = 2\ \mathrm{atm}$.)

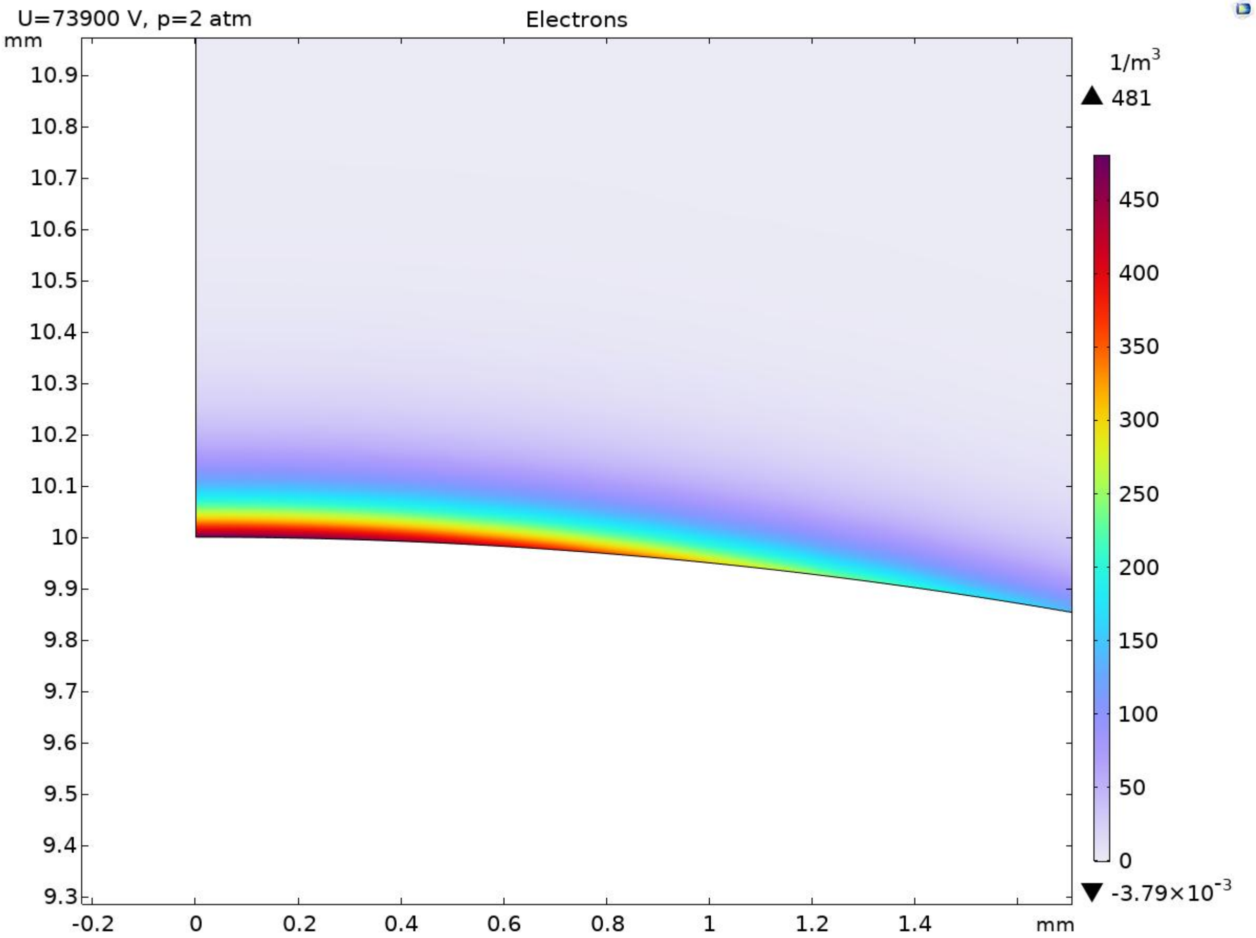


Fig. 18: Zoom of Fig. 17.

The Resonance Method

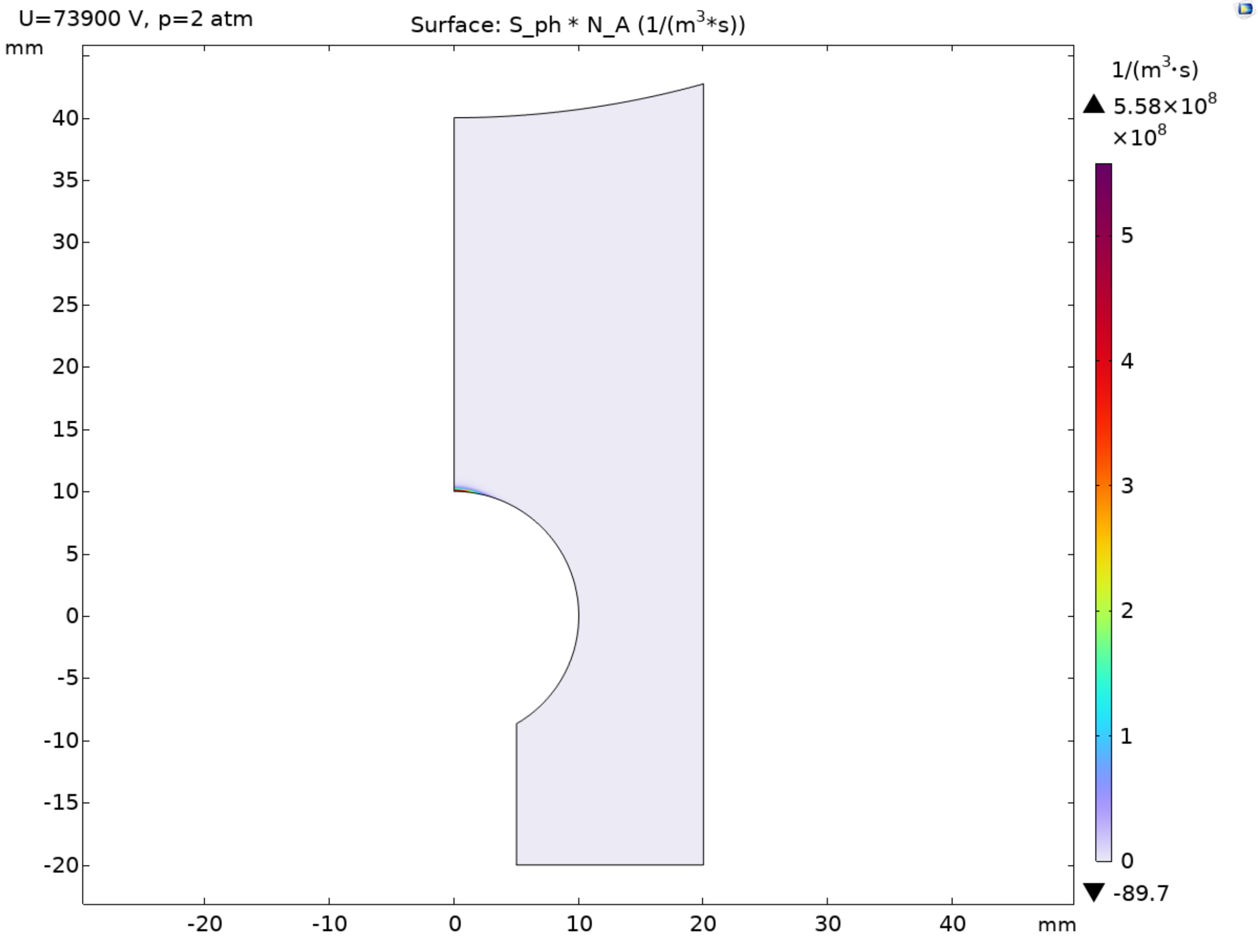


Fig. 19: Distribution of the photoionization rate for voltage just below the resonance point for air, $d = 30$ mm – positive (physical) values, created using [1]. (Note that $p = 2$ atm.)

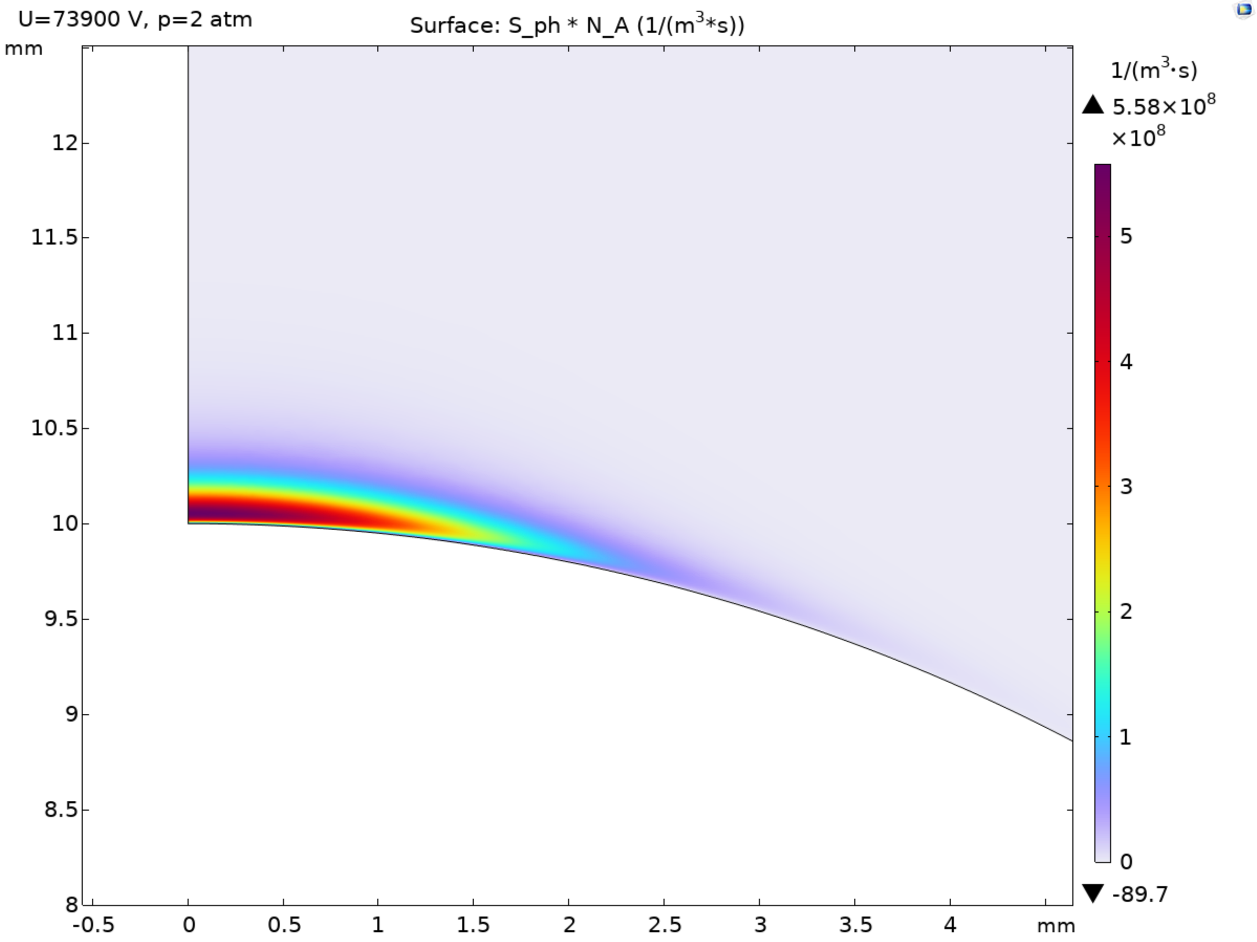


Fig. 20: Zoom of Fig. 19.

The Resonance Method

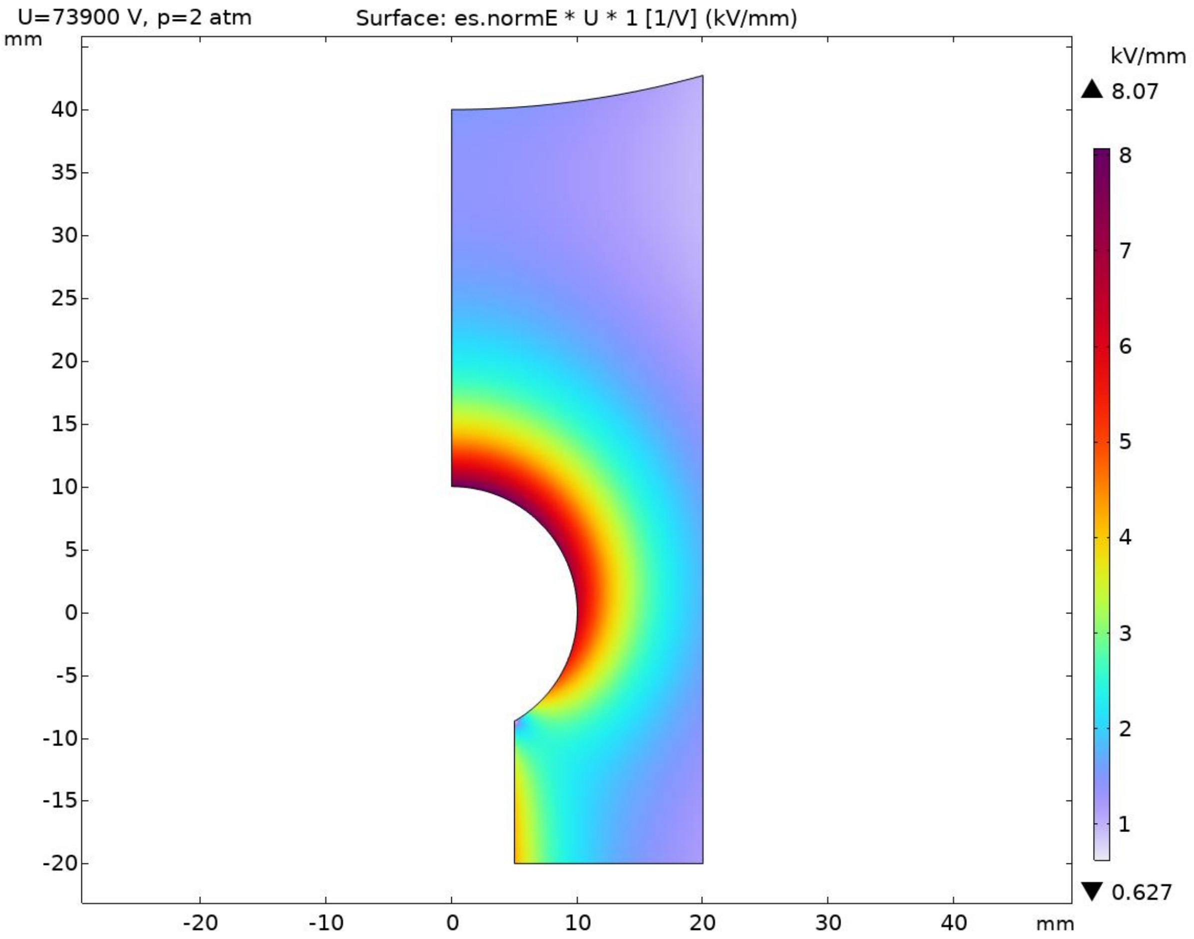


Fig. 21: Electric field intensity distribution for voltage just below the resonance point for air, $d = 30\ \mathrm{mm}$, created using [1]. (Note that $p = 2\ \mathrm{atm}$.)

The Resonance Method

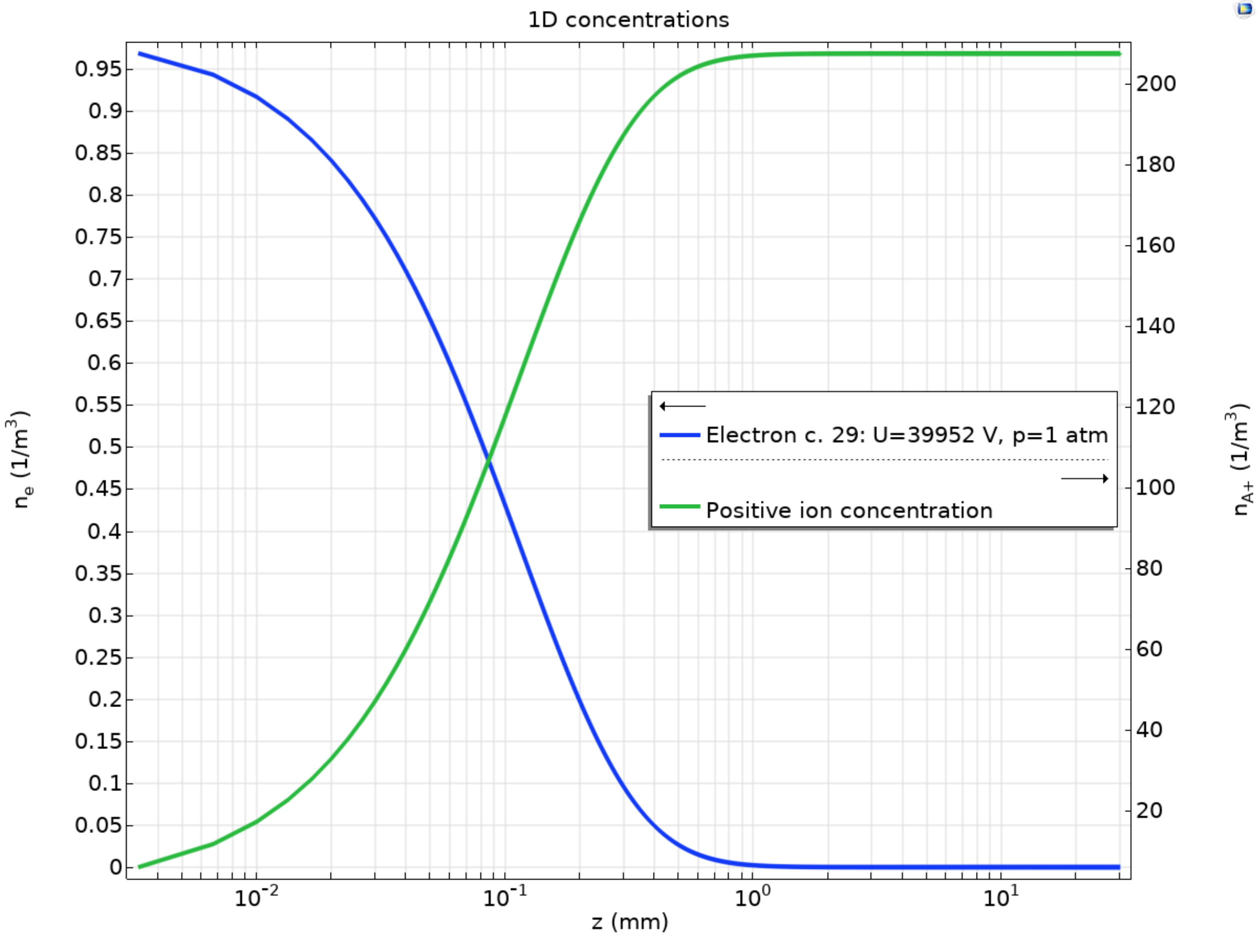


Fig. 22: Concentration distribution of electrons and positive ions along the axis of symmetry for voltage just below the resonance point for air, $d = 30$ mm and $p = 1$ atm – positive (physical) values, created using [1].

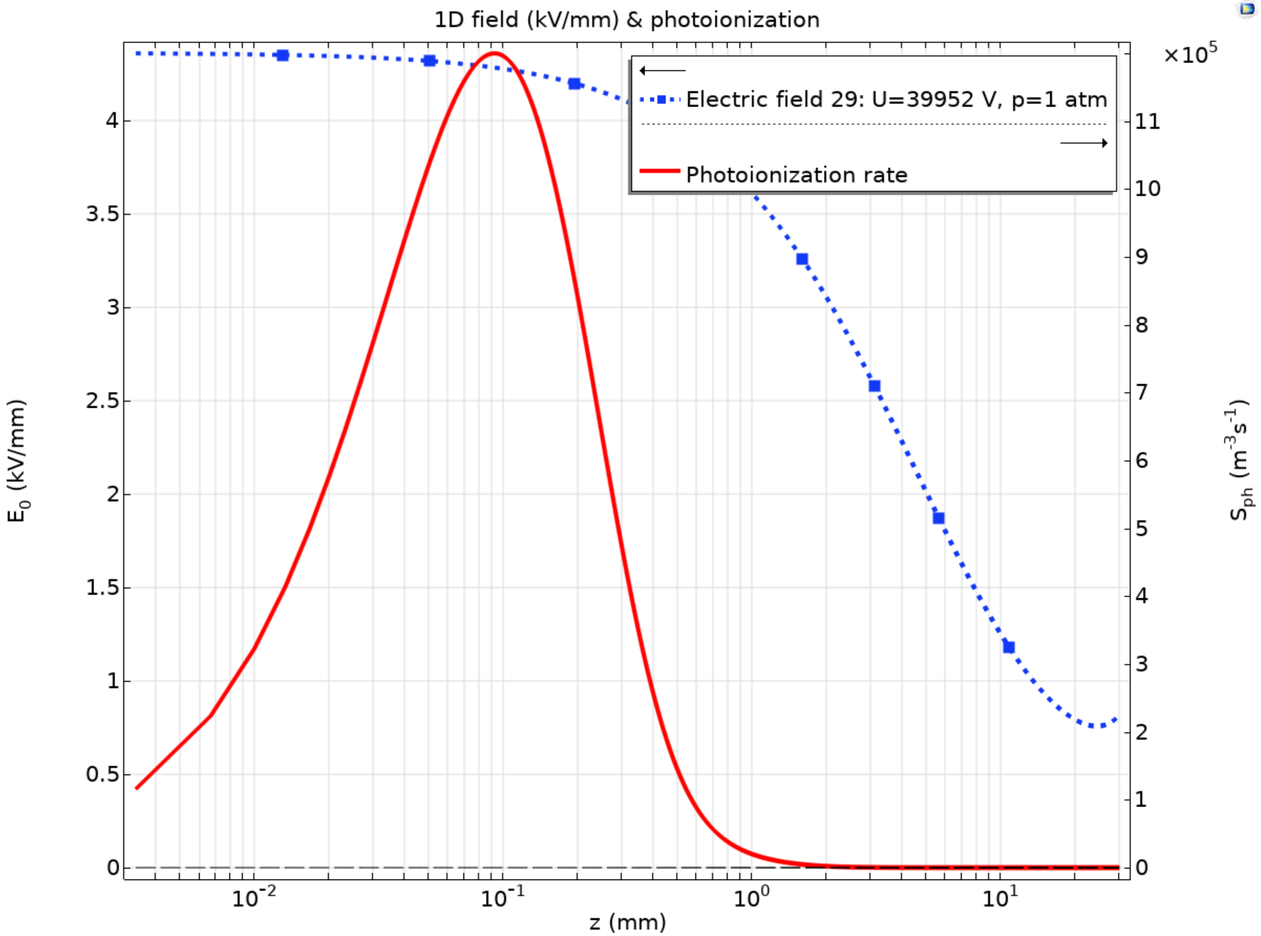


Fig. 23: Distribution of S_{ph} and E along the axis of symmetry for voltage just below the resonance point for air, $d = 30$ mm and $p = 1$ atm , created using [1].

The Resonance Method

- The corona stabilization effects cause a higher breakdown voltage above the calculated U_0
- Corona discharge is localized self-sustaining discharge without breakdown
- To calculate breakdown voltage, nonlinear processes must be added:
  - Poisson (with a non-zero right-hand side) and drift-diffusion-reaction equations coupling
  - Nonlinear reactions

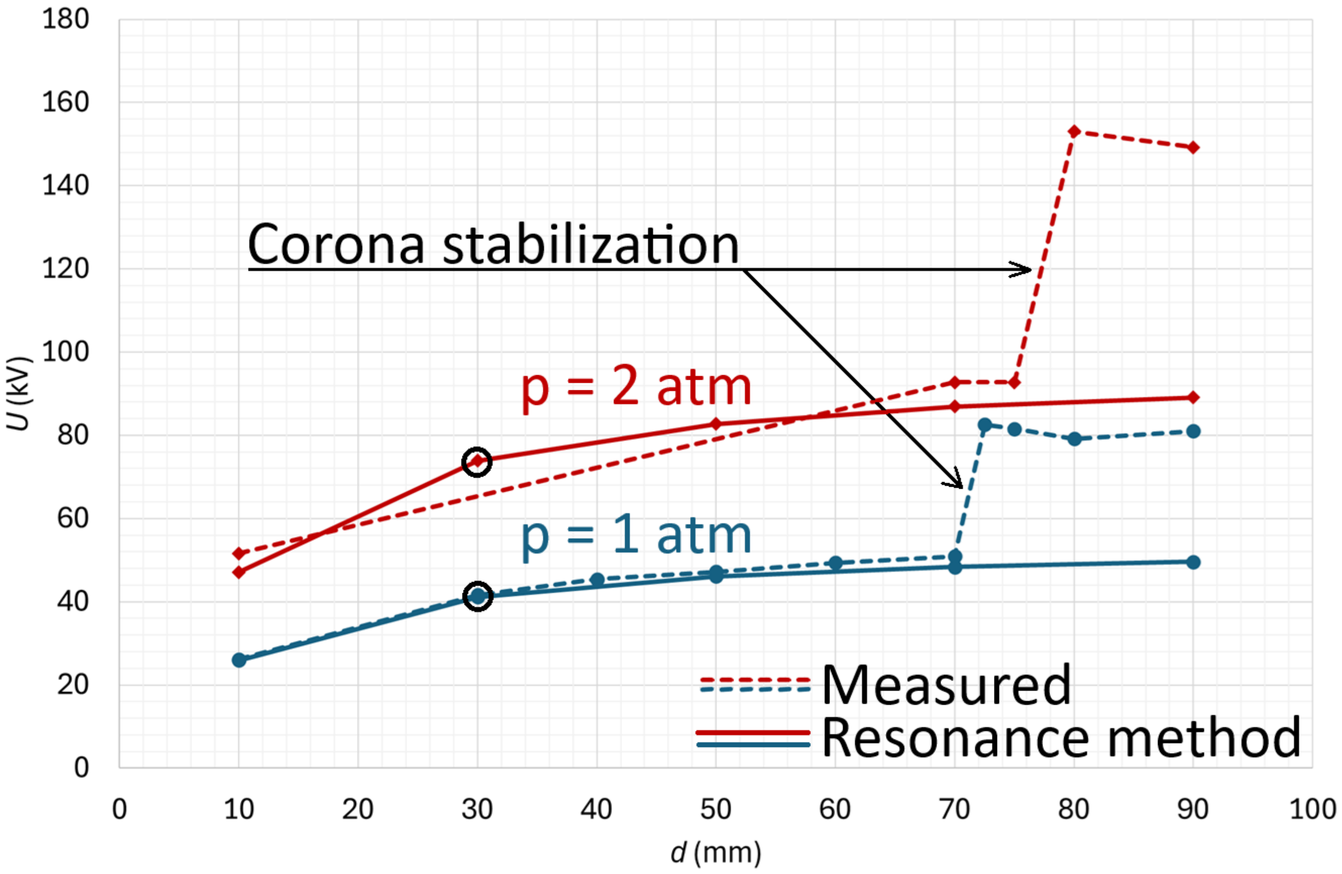


Fig. 24: Calculated self-sustainment voltages U_0 using the resonance method (solid curves) and measured breakdown voltages (dashed curves) as a function of gap distance d for $R_{\mathrm{Sphere}} = 75\ \mathrm{mm}$ and $r_{\mathrm{Sphere}} = 10\ \mathrm{mm}$ and air at $p = 1\ \mathrm{atm}$ (blue) and $p = 2\ \mathrm{atm}$ (red).

Thank you for your attention

Ing. Filip Zmeko
zmekof@fel.zcu.cz

This work was supported by SGS-2024-014 project.

Konference COMSOL Multiphysics 2026
May 29, 2026

[1] COMSOL Multiphysics® v. 6.1. www.comsol.com. COMSOL AB, Stockholm, Sweden.
[2] MATLAB, 2023. MATLAB Version: 9.8.0.1359463 (R2020a). Natick, Massachusetts: The MathWorks Inc.
[3] Ferreira, N. G. C., et al. "Simulation of pre-breakdown discharges in high-pressure air. I: The model and its application to corona inception." Journal of Physics D: Applied Physics 52.35 (2019): 355206.
[4] G.J.M. Hagelaar and L.C. Pitchford, "Solving the Boltzmann equation to obtain electron transport coefficients and rate coefficients for fluid models", Plasma SciSources and Tech 14, 722 (2005).
[5] SERDYUK, Yuriy V. Propagation of cathode-directed streamer discharges in air. In: Proc. COMSOL Conf., Rotterdam, The Netherlands. 2013.
[6] BOURDON, Anne, et al. Efficient models for photoionization produced by non-thermal gas discharges in air based on radiative transfer and the Helmholtz equations. Plasma Sources Science and Technology, 2007, 16.3: 656.
[7] Benilov, M. S., et al. "A practical guide to modeling low-current quasi-stationary gas discharges: Eigenvalue, stationary, and time-dependent solvers." Journal of Applied Physics 130.12 (2021).