A SIMPLE NUMERICAL METHOD FOR CALCULATION OF DIFFRACTION INTEGRALS

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Abstract

In practice, it is necessary to calculate the point-spread function and the optical transfer function of optical systems in order to evaluate the quality of these systems. The two-dimensional integrals are often calculated over the region, which cannot be analytically expressed, and the integrand is a complicated function. For mentioned reasons the integrals cannot be evaluated explicitly and suitable techniques for numerical integration must be used. A large number of such calculations must be carried out during the optimisation of parameters of the optical system. The computing time is relatively long in case of evaluation of a large number of integrals. Our work describes a simple method for numerical calculation of diffraction integrals, which offers a suitable accuracy and reduces substantially the computing time.

1. Introduction

Two basic approaches are used for evaluation of the image quality of optical systems. The first is based on the theory of geometrical optics [1,2,3] and the latter on the theory of wave optics [1,2,3]. A basic characteristic of the optical system is the point-spread function [1,2,3], which describes the distribution of the intensity of light in the image of a point source. It is very well known that the imaging quality depends on the aberrations of the optical system [1,2,3] and subsequently the point-spread function also depends on these aberrations. Moreover, it will also be affected by the shape of the pupil of the optical system. The point-spread function is expressed by the diffraction integral over the area of the optical system and the integrand depends on the aberrations and the transmissivity of the optical system. Since the two-dimensional diffraction integrals are evaluated over the region, which in most cases cannot be expressed analytically, and the integrand is often a very complicated function, they cannot be calculated explicitly. It is necessary to use some suitable technique for numerical integration [4,5]. It is required to know the point-spread functions across the whole image plane of the optical system in order to obtain a good view of the quality of the optical system. For common optical systems, e.g. camera lenses, this represents a calculation of several thousands of two-dimensional integrals. Such calculations are relatively time consuming. As we can see from the description of the problem it is desirable to use such methods of numerical integration, which give us in the shortest time sufficiently accurate results. The presented work describes such method for numerical calculation, which was implemented in Matlab.

2. Diffraction integral

Consider a scalar wave field, which is described by the scalar function $V(M,t)$. It is known from the theory of electromagnetic field [1,2,3] that the function $V(M,t)$ satisfies the wave equation

$$\nabla^2 V(M,t) = \frac{1}{v^2}\frac{\partial^2 V(M,t)}{\partial t^2} \quad , \tag{1}$$

where M is some point of the wave field, t denotes time, v is the velocity of the propagation of waves and ∇^2 is the Laplace operator [5]. Suppose now the solution of the wave equation has the form

$$V(M,t) = U(M)e^{-i\omega t}$$

where $\omega = 2\pi\nu$, and ν is the frequency. Function $U(M)$ is then a solution of Helmholtz equation. Solving Helmholtz equation by Green's function method [1, 4] we obtain integral

$$U(P) = -\frac{i}{\lambda}\iint_S U(M)\frac{e^{ikr}}{r}\cos(n,r)\,\mathrm{d}S \ , \tag{2}$$

where r is the distance between points M and P, $\cos(n,r)$ is the cosine of the angle between the normal n and the direction of r. From this relation one can determine the field amplitude $U(P)$ in an arbitrary point P of the area limited by the surface S, if the field amplitude $U(M)$ is known on the surface S. We will now calculate the integral (2) for the optical system with the aberrations. Using Fraunhofer approximation the equation (2) has the form [1]

$$U(s,t) = K\iint_S F(p,q)e^{-2\pi i(ps+qt)}\,\mathrm{d}p\,\mathrm{d}q \ , \tag{3}$$

where K is constant. We can determine the amplitude of the wave field in the image plane of the optical system with a finite numerical aperture using Eq.(3). From the mentioned equation it is clear that field $U(s,t)$ is proportional to the Fourier transform of function $F(p,q)$. In practice, it is most important to know the field $U(0,0)$. For common optical systems with a small numerical aperture and uniformly transmissive pupil, e.g. camera lenses, the previous equation simplifies to the form

$$U(0,0) = K\iint_S \exp\left[2\pi i W(x,y)/\lambda_o\right]\mathrm{d}x\mathrm{d}y \, , \tag{4}$$

where $W(x,y)$ is the wavefront aberration of the optical system, λ_o is the wavelength of light in the vacuum [1].

3. Numerical calculation of diffraction integral

As described above, we are interested in calculation of diffraction integrals, which have the following form

$$I = \iint_S \exp[2\pi i f(x,y)]\mathrm{d}x\mathrm{d}y = C + iS \, , \tag{5a}$$

where

$$C = \iint_S \cos 2\pi f(x,y)\,\mathrm{d}x\mathrm{d}y \qquad \text{and} \qquad S = \iint_S \sin 2\pi f(x,y)\,\mathrm{d}x\mathrm{d}y\,. \tag{5b}$$

These integrals can be calculated with very well known methods for numerical integration [5], such as for example the rectangular rule, trapezoidal rule, Simpson's method, Newton-Cotes method, Gauss quadrature, etc. However, these methods need to evaluate functions $\sin[2\pi f(x,y)]$ and $\cos[2\pi f(x,y)]$ at many grid points. We can modify the rectangular rule in order to eliminate such calculations.

We make an equidistant grid on the two-dimensional region of integration and for the centre of every cell with indices (i,j), where $i = 1,2,...,N_x$ and $j = 1,2,...,N_y$, we calculate the value of the function $f(x_i,y_j) = f(i,j)$. From obtained values we take only the fractional part $g(i,j)$. If this fractional part is negative, then we add one to the value $g(i,j)$, i.e.

$$g(i,j) = g(i,j) \quad \text{for } g > 0, \qquad\qquad g(i,j) = g(i,j) + 1 \quad \text{for } g < 0. \tag{6}$$

The described transformation of the function $g(i,j)$ is given by the periodicity of trigonometric functions $\sin f(x,y)$ and $\cos f(x,y)$, i.e.

$$\sin 2\pi g = \sin 2\pi[k+g] = \sin 2\pi[k+(g+1)]\,,$$
$$\cos 2\pi g = \cos 2\pi[k+g] = \cos 2\pi[k+(g+1)]\,,$$

where $k = 0, 1, 2, ...$ The transformed values $g(i,j)$ are located in the range <0,1>. We will round the values $g(i,j)$, e.g. with the accuracy $\delta g = 0.025$. The described integrals are given by

$$S = \frac{\sum_{k=0}^{d} N_k \sin[2\pi k/d]}{N} A\,, \qquad C = \frac{\sum_{k=0}^{d} N_k \cos[2\pi k/d]}{N} A\,, \tag{7}$$

where $d = 1/(2\delta g)$, N_k is the number of rounded values $g(i,j)$, which are equal to k/d, $N = N_x N_y$ is the overall number of working cells, and A is the area of integration. The accuracy $\delta g = 0.025$ was not chosen arbitrarily, but it corresponds to the change of the wave aberration $\delta W=\lambda/40$. Such small change of the wave aberration has practically no influence on the point-spread function and even from the experimental point of view it is hardly to measure. The described technique can be generalized for an arbitrary chosen value δg. We can also see that it is not necessary to evaluate functions sine and cosine to calculate the integrals. This fact speeds up the calculation, which is very important in practice. Further improving of the computing time and accuracy of the proposed method can be obtained by calculating values $f(i,j)$ on a coarse grid first and then interpolating the values on a dense grid.

As an example of the mentioned technique, we show the calculation of such integrals. The wavefront aberration W can be expressed as

$$W(x,y) = W_{20}(x^2+y^2) + W_{31}(x^2+y^2)x\,, \tag{8}$$

where W_{20} is the coefficient of defocusing, and W_{31} is the coefficient of the third order coma. The integrals take the form

$$S(W) = \int_0^1\int_0^1 \sin\left[2\pi W\right] \mathrm{d}x\mathrm{d}y \quad , \quad C(W) = \int_0^1\int_0^1 \cos\left[2\pi W\right] \mathrm{d}x\mathrm{d}y \ , \qquad (9)$$

We compared the accuracy of the presented technique with the method for numerical integration in Matlab. Let $N_x = N_y = 30$, then the calculated integrals (9) are shown in table 1 for different values of coefficients W_{20} and W_{31}. *S*, *C* are the values calculated with the described method, S_2,C_2 are the values calculated with the *quad2* function in Matlab. We can see that the accuracy is sufficient even in case of a coarse approximation. The computing time of the method was several times shorter than for the method *quad2*. If we need a better accuracy then the number of working cells must be larger.

Table 1

Method	$W_{20} = 0.1$ $W_{31} = 0.25$	$W_{20} = 0.25$ $W_{31} = 0.25$	$W_{20} = 0.25$ $W_{31} = 0.5$
S	0.5315	0.4345	0.2963
C	0.3838	0.0971	0.1129
S_2	0.5544	0.4614	0.3044
C_2	0.3981	0.0955	0.0983

4. Conclusion

A simple technique for calculation of this diffraction integral was proposed. The technique does not evaluate values of the integrand as necessary in case of known methods of numerical integration. A numerical calculation of the integral is therefore substantially faster than the common methods and the accuracy is sufficient for practice. The proposed technique can be also used for calculation of the optical transfer function of the optical system. The method can be simply generalized and one can obtain a predefined accuracy. The technique can be applied both in optics and in other branches of physics and engineering.

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References

[1] A.Mikš, Aplikovaná optika, Vydavatelství ČVUT, Praha 2000.
[2] M.Born, E.Wolf, Principles of Optics. Oxford University Press, New York, 1964.
[3] M.V.Klein: Optics. John Wiley & Sons, Inc., New York, 1970.
[4] V.J.Arsenin: Matematičeskaja fizika. Nauka, Moskva, 1966.
[5] K.Rektorys: Přehled užité matematiky. SNTL, Praha, 1968.

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