SYMBOLIC ANALYSIS OF LINEAR ELECTRIC CIRCUITS

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Abstract

In present days there exist lots of programs such as PSPICE, TINA, which enable to solve circuits numerically. But sometimes it is useful to use program which offers symbolic results for solved circuit. The paper is devoted to the creation of application program for symbolic analysis of linear circuit system containing independent and dependent sources of DC voltage and/or current. This program is based on nodal analysis.

1 Principles of Nodal Analysis and its Matrix Formulation

Nodal analysis is method for solving linear electric circuits. Unknown quantities are nodal voltages. A nodal voltage is a voltage between a node pair, which is formed by a nonference node and a reference node. A node is a point of connection of two or more circuit elements. A word formulation of nodal analysis procedure is following [1]:

The first step is determination the number of nodes in the circuit. One of the nodes is selected as the reference node and a node voltage is assigned to each nonreference node. For an N_u-node circuit there are $N_\mathrm{u}-1$ nodal voltages, so $N_\mathrm{u}-1$ linearly independent equations are required to solve for the nodal voltages.

The second step is a writing a constraint equation for each independent or dependent voltage source in the circuit in terms of the assigned nodal voltages using Kirchhoff's voltage law (KVL). If the *k*th voltage source $u_k^{(\mathrm{S})}$ is connected between a nonreference node *p* and the reference node, then the nodal voltage $u_p^{(\mathrm{n})}$ is given by source voltage accounting for the polarities, and the constraint equation takes the following form:

$$u_p^{(\mathrm{n})} = \pm u_k^{(\mathrm{S})} \tag{1}$$

If the *k*th voltage source $u_k^{(\mathrm{S})}$ is connected between a nonreference node *p* and a nonreference node *q*, then the difference of the nodal voltages $u_p^{(\mathrm{n})}$ and $u_q^{(\mathrm{n})}$, assigned to these nonreference nodes, is given by source voltage accounting for the polarities, and the constraint equation takes the form:

$$u_p^{(\mathrm{n})} - u_q^{(\mathrm{n})} = \pm u_k^{(\mathrm{S})} \tag{2}$$

Each constraint equation represents one of the required linearly independent equations. For each dependent voltage source it is necessary to express the controlling variable in terms of the nodal voltages. For a circuit containing N_NZ voltage sources, with $N_\mathrm{NZ}^{(\mathrm{1nonref})}$ connected between a nonreference node and the reference node and $N_\mathrm{NZ}^{(\mathrm{2nonref})}$ connected between two nonreference nodes, a number of the constraint equations is $N_\mathrm{NZ} = N_\mathrm{NZ}^{(\mathrm{1nonref})} + N_\mathrm{NZ}^{(\mathrm{2nonref})}$ and these equations yield N_NZ linearly independent equations.

The remaining $N_\mathrm{u}-1-N_\mathrm{NZ}$ linearly independent equations must be formulated by applying Kirchhoff's current law (KCL) at each of $N_\mathrm{u}-1-N_\mathrm{NZ}-N_\mathrm{NZ}^{(\mathrm{2nonref})}$ nonreference nodes not connected to a voltage source, and at each of $N_\mathrm{NZ}^{(\mathrm{2nonref})}$ supernodes, in case that only one voltage source is connected to each node. A supernode is a closed surface that bounded two nonreference nodes connected by a voltage source.

Because MATLAB (MATrix LABoratory) is a tool for matrix solving of problems, it is useful to create a matrix formulation of nodal analysis.

We assume a proper linear electric circuit containing N_{u} nodes and N_{v} branches. Let us consider that a resistor with resistance $R_k \neq 0 \quad \wedge \quad R_k \neq \infty$, an ideal current source (independent or dependent) with a current $i_k^{(\mathrm{S})}$ and an ideal voltage source with a voltage $u_k^{(\mathrm{S})}$ are connected in a branch v_k $(k = 1, 2, ..., n_{\mathrm{v}})$ with nodes *p* and *q* (Figure 1).

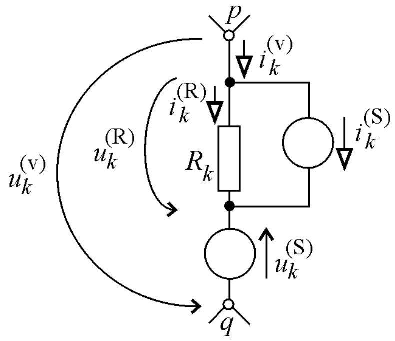

Figure 1: The *k*th circuit branch connected to the node *p* and the node *q*

KCL written in the matrix form is [2]:

$$\mathbf{A}\,\mathbf{I}^{(\mathrm{v})} = \mathbf{0} \tag{3}$$

where $\mathbf{I}^{(\mathrm{v})}$ is a matrix of branch currents $i_k^{(\mathrm{v})}$ of order $(N_{\mathrm{v}}, 1)$, $\mathbf{A}$ being the incidence matrix of order $(N_{\mathrm{u}} - 1, N_{\mathrm{v}})$ - a matrix with elements a_{mn} which express incidence between nodes and branches of circuit and their value is $\{+1, -1, 0\}$.

The branch current $\mathbf{I}^{(\mathrm{v})}$ can be expressed (Figure 1):

$$\mathbf{I}^{(\mathrm{v})} = \mathbf{I}^{(\mathrm{R})} + \mathbf{I}^{(\mathrm{S})} \tag{4}$$

where $\mathbf{I}^{(\mathrm{R})}$ is a matrix of resistor currents $i_k^{(\mathrm{R})}$ of order $(N_{\mathrm{v}}, 1)$, $\mathbf{I}^{(\mathrm{S})}$ being a matrix of current source currents $i_k^{(\mathrm{S})}$ of order $(N_{\mathrm{v}}, 1)$.

The resistor current $\mathbf{I}^{(\mathrm{R})}$ will be:

$$\mathbf{I}^{(\mathrm{R})} = \mathbf{G}\left(\mathbf{U}^{(\mathrm{v})} + \mathbf{U}^{(\mathrm{S})}\right) \tag{5}$$

where $\mathbf{G}$ is a diagonal matrix of resistor conductance of order $(N_{\mathrm{v}}, N_{\mathrm{v}})$, $\mathbf{U}^{(\mathrm{v})}$ being a matrix of branch voltages of order $(N_{\mathrm{v}}, 1)$, $\mathbf{U}^{(\mathrm{S})}$ being a matrix of voltage source voltages of order $(N_{\mathrm{v}}, 1)$.

Each branch voltage can be expressed in terms of a nodal voltage [1]:

$$\mathbf{U}^{(\mathrm{v})} = \mathbf{A}^{\mathrm{T}}\mathbf{U}^{(\mathrm{n})} \tag{6}$$

where $\mathbf{A}^{\mathrm{T}}$ is a transpose matrix to a matrix $\mathbf{A}$, $\mathbf{U}^{(\mathrm{n})}$ being a matrix of nodal voltages of order $(N_{\mathrm{u}} - 1, 1)$.

Substituting equations (4), (5) and (6) into equation (3), we obtain:

$$\mathbf{A}\,\mathbf{G}\mathbf{A}^{\mathrm{T}}\mathbf{U}^{(\mathrm{n})} + \mathbf{A}\mathbf{G}\mathbf{U}^{(\mathrm{S})} + \mathbf{A}\mathbf{I}^{(\mathrm{S})} = \mathbf{0} \tag{7}$$

The equation (7) represents a formulation of nodal equations in terms of incidence matrix $\mathbf{A}$ for electric circuits containing current and/or voltage sources (independent and dependent).

Matrix $\mathbf{U}^{(\mathrm{S})}$ in the equation (7) takes the following form:

$$\mathbf{U}^{(\mathrm{S})} = \mathbf{U}^{(\mathrm{IVS})} + \mathbf{U}^{(\mathrm{CCVS})} + \mathbf{U}^{(\mathrm{VCVS})} \tag{8}$$

where $\mathbf{U}^{(\mathtt{IVS})}$ is a matrix of voltages independent voltage sources with a positive sign if its polarity is not oriented corresponding to a branch orientation, $\mathbf{U}^{(\mathtt{CCVS})}$ being a matrix of voltages of current-controlled voltage sources (CCVS), $\mathbf{U}^{(\mathtt{VCVS})}$ being a matrix of voltages of voltage-controlled voltage sources (VCVS).

Matrix $\mathbf{U}^{(\mathtt{CCVS})}$ in the equation (8) is following:

$$\mathbf{U}^{(\mathtt{CCVS})} = \mathbf{V}^{(\mathtt{CCVS})}\mathbf{I}^{(\mathtt{v})} \tag{9}$$

where $\mathbf{V}^{(\mathtt{CCVS})}$ is a matrix of coupling coefficients between voltages CCVS and controlling currents with a positive sign if its polarity is not oriented corresponding to an orientation of branch in which is the source, and simultaneously a polarity of controlling current is corresponding to an orientation of branch in which it is placed.

Matrix $\mathbf{U}^{(\mathtt{VCVS})}$ in the equation (8) is following:

$$\mathbf{U}^{(\mathtt{VCVS})} = \mathbf{V}^{(\mathtt{VCVS})}\mathbf{U}^{(\mathtt{v})} \tag{10}$$

where $\mathbf{V}^{(\mathtt{VCVS})}$ is a matrix of coupling coefficients between voltages VCVS and controlling voltages with a positive sign if its polarity is not oriented corresponding to an orientation of branch in which is the source, and simultaneously a polarity of controlling voltage is oriented corresponding to an orientation of branch in which it is placed.

Matrix $\mathbf{I}^{(\mathtt{S})}$ in the equation (7) is following:

$$\mathbf{I}^{(\mathtt{S})} = \mathbf{I}^{(\mathtt{ICS})} + \mathbf{I}^{(\mathtt{CCCS})} + \mathbf{I}^{(\mathtt{VCCS})} \tag{11}$$

where $\mathbf{I}^{(\mathtt{ICS})}$ is a matrix of currents independent current sources with a positive sign if its polarity is oriented corresponding to a branch orientation, $\mathbf{I}^{(\mathtt{CCCS})}$ being a matrix of currents of current-controlled current sources (CCCS), $\mathbf{I}^{(\mathtt{VCCS})}$ being a matrix of currents of voltage-controlled current sources (VCCS).

Matrix $\mathbf{I}^{(\mathtt{CCCS})}$ in the equation (11) is following:

$$\mathbf{I}^{(\mathtt{CCCS})} = \mathbf{V}^{(\mathtt{CCCS})}\mathbf{I}^{(\mathtt{v})} \tag{12}$$

where $\mathbf{V}^{(\mathtt{CCCS})}$ is a matrix of coupling coefficients between currents CCCS and controlling currents with a positive sign if its polarity is oriented corresponding to an orientation of branch in which is the source, and simultaneous a polarity of controlling current is oriented corresponding to an orientation of branch in which it is placed.

Matrix $\mathbf{I}^{(\mathtt{VCCS})}$ in the equation (11) is following:

$$\mathbf{I}^{(\mathtt{VCCS})} = \mathbf{V}^{(\mathtt{VCCS})}\mathbf{U}^{(\mathtt{v})} \tag{13}$$

where $\mathbf{V}^{(\mathtt{VCCS})}$ is a matrix of coupling coefficients between currents VCCS and controlling currents with a positive sign if its polarity is oriented corresponding to an orientation of branch in which is the source, and simultaneous a polarity of controlling current is oriented corresponding to an orientation of branch in which it is placed.

Substituting equations (8) up to (13) into equation (7) we obtain the matrix formulation of nodal equations for linear circuits having independent and/or dependent sources:

$$\mathbf{G}^{*(\mathtt{n})}\mathbf{U}^{(\mathtt{n})} = \mathbf{I}^{(\mathtt{n})} \tag{14}$$

where $\mathbf{G}^{*(\mathtt{n})}$ is a node conductance matrix of order $(N_{\mathrm{u}} - 1, N_{\mathrm{u}} - 1)$, $\mathbf{I}^{(\mathtt{n})}$ being a column matrix of short-circuiting currents produced by the current sources and voltage sources of order $(N_{\mathrm{u}} - 1, 1)$.

- The obtained system of the nodal equations (14) must be treated for ideal current and voltage sources by following way.

An internal resistance of an ideal current source that is connected in the *k*th circuit branch is infinitely large and so its internal conductance is equal zero ($g_{kk} \to 0$).

An internal resistance of an ideal voltage source that is connected in the *k*th circuit branch is equal zero and so its internal conductance is infinitely large ($g_{kk} \to \infty$).

For that reason it is necessary to arrange the system of nodal equation (14). If a voltage source is connected between nonreference node *p* and reference node, a limit for $g_{kk} \to \infty$ of the left-hand side of equation written for nonreference node *p* must be found

$$\lim_{g_{kk}\to\infty} \left(\sum_{j=1}^{N_{\mathrm{u}}-1} \frac{g_{pj}^{*(\mathrm{n})}}{g_{kk}} u_j^{(\mathrm{n})} \right) \tag{15}$$

as well as a limit for $g_{kk} \to \infty$ of right-hand side of equation written for nonreference node *p*

$$\lim_{g_{kk}\to\infty} \frac{i_p^{(\mathrm{n})}}{g_{kk}} \tag{16}$$

and the constraint equation in the form (1) will be obtained by this arrangement.

If a voltage source is connected between a nonreference node *p* and a nonreference node *q*, a limit for $g_{kk} \to \infty$ of the both sides of equation written for nonreference node *p* must be found and the constraint equation in the form (2) will be obtained by this arrangement; but one more equation is needed, which will be provided by a nodal equation for supernode $p-q$ in such a way, that we add the equation for nonreference node *p* and the equation for nonreference node *q*

$$\sum_{j=1}^{N_{\mathrm{u}}-1} \left(g_{pj}^{*(\mathrm{n})} + g_{qj}^{*(\mathrm{n})} \right) u_j^{(\mathrm{n})} = i_p^{(\mathrm{n})} + i_q^{(\mathrm{n})} \tag{17}$$

If any coefficient $g_{pj}^{*(\mathrm{n})}$ in the equation (17) includes a term $\pm g_{kk}$ (independent voltage source) or a term $\pm \rho_k g_{kk} g_{ll}$ (CCVS) or a term $\pm \alpha_k g_{kk}$ (VCVS), then a coefficient $g_{qj}^{*(\mathrm{n})}$ includes a term $\mp g_{kk}$ or a term $\mp \rho_k g_{kk} g_{ll}$ or a term $\mp \alpha_k g_{kk}$, and there is no any coefficient with term g_{kk} in the equation (17).

2 Results

Two selected circuits will be presented as example of nodal analysis by using the program that generates the system of symbolic nodal equations. The first one is a circuit having independent sources only (Figure 1 left) and the second one is a circuit having independent and dependent sources (Figure 1 right).

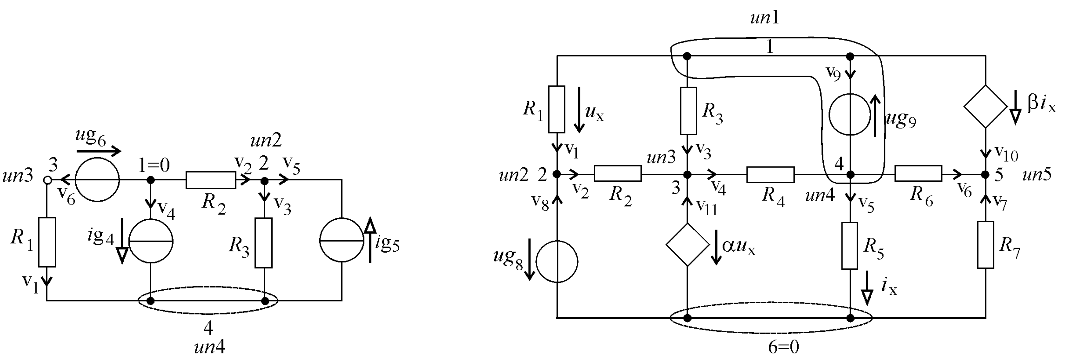

Figure 1: An example of circuit having independent sources only (left) and a circuit having dependent sources (right)

After assigning an input data (parameters of the circuit elements, the incidence matrix and the reference node) the program generates a system of symbolic nodal equations. This system is subsequently solved and the nodal voltages are obtained in numeric form. The branch voltages and the branch currents are obtained in both symbolic and numeric form.

Referring to Figure 1, the program generates the system of equations (Figure 2 left), which consists of one constraint equation and two nodal equations for non reference nodes 2 and 4. For given values of circuit parameters ($R_1 = 20\Omega, R_2 = 4\Omega, R_3 = 12\Omega, ig_4 = 0.3\mathrm{A}, ig_5 = 0.9\mathrm{A}, ug_6 = 12\mathrm{V}$) the program computes numeric values of nodal voltages (Figure 2 right).

```
Constraint equations:

      un3  =  ug6

Nodal equations for non reference nodes:

  node 2:  (g2+g3)*un2-g3*un4-ig5   =   ig5

  node 4:  -un2*g3-g1*un3+(g1+g3)*un4-ig4+ig5   =   ig4-ig5
```

```
Nodal voltages - numeric values:

  un2 =  3.200000   V

  un3 = 12.000000   V

  un4 =  2.000000   V
```

Figure 2: Symbolic equations generated by program for circuit having independent sources (left) and numeric values of nodal voltages (right)

The obtained results of the branch voltages (Figure 3 left) and the branch currents (Figure 3 right) for circuit having independent sources and the values of circuit element parameters are given in Figure 3.

```
Branch voltages:

  uv1 = un3-un4 =  10.000000  V

  uv2 = -un2 =  -3.200000  V

  uv3 = un2-un4 =   1.200000  V

  uv4 = -un4 =  -2.000000  V

  uv5 = un2-un4 =   1.200000  V

  uv6 = -un3 = -ug6 = -12.000000  V
```

```
Branch currents:

  iv1 = (un3-un4)*g1 =  0.500000   A

  iv2 = -g2*un2 = -0.800000   A

  iv3 = (un2-un4)*g3 =  0.100000   A

  iv4 = ig4 =  0.300000   A

  iv5 = -ig5 = -0.900000   A

  iv6 = iv1 =  0.500000   A
```

Figure 3: The branch voltages (left) and the branch currents (right) for circuit having independent sources

Referring to Figure 1 (right), the program generates the system of equations (Figure 4 left), which consists of three constraint equations and two nodal equations, one of them is for a nonreference node 5 and the other one is for a supernode 1-4. For given values of circuit parameter($R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = R_7 = 1\mathrm{k}\Omega$, $ug_8 = 12\mathrm{V}, ug_9 = 6\mathrm{V}, ig_{10} = \beta i_x = 2 i_x \mathrm{A}$, $ug_{11} = \alpha u_x = 2 u_x \mathrm{V}$) the program computes the following numeric values of nodal voltages (Figure 4 right):

```
Constraint equations:

      un2  =  ug8

      -alfa1*un1+alfa1*un2+un3  =  0

      -un1+un4  =  ug9

Nodal equations for non reference nodes:

  node 5:  (-beta1*g5-g6)*un4+(g6+g7)*un5  =  0

Nodal equations for supernodes:

  supernode 1-4:  (g1+g3)*un1-g1*un2+(-g3-g4)*un3+(beta1*g5+g4+g5+g6)*un4-g6*un5  =  0
```

```
Nodal voltages - numeric values:

  un1 =-38.000000   V

  un2 = 12.000000   V

  un3 =-100.000000   V

  un4 =-32.000000   V

  un5 =-48.000000   V
```

Figure 4: Symbolic equations generated by program for circuit having dependent sources (left) and numeric values of nodal voltages (right)

The obtained results of the branch voltages (Figure 5 left) and the branch currents (Figure 5 right) for circuit having dependent sources and the given values of circuit element parameters are in Figure 5.

```
Branch voltages:

uv1 = un1-un2 = -50.000000   V
uv2 = un2-un3 = 112.000000   V
uv3 = un1-un3 =  62.000000   V
uv4 = un3-un4 = -68.000000   V
uv5 = un4 = -32.000000   V
uv6 = un4-un5 =  16.000000   V
uv7 = -un5 =  48.000000   V
uv8 = -un2 = -ug8 = -12.000000   V
uv9 = un1-un4 = -ug9 =  -6.000000   V
uv10 = un1-un5 =  10.000000   V
uv11 = -un3 = -2*uv1 = 100.000000   V
```

```
Branch currents:

iv1 = (un1-un2)*g1 = -0.050000   A
iv2 = (un2-un3)*g2 =  0.112000   A
iv3 = (un1-un3)*g3 =  0.062000   A
iv4 = (un3-un4)*g4 = -0.068000   A
iv5 = un4*g5 = -0.032000   A
iv6 = (un4-un5)*g6 =  0.016000   A
iv7 = -un5*g7 =  0.048000   A
iv8 = -iv1+iv2 =  0.162000   A
iv9 = -iv1-iv3-iv10 =  0.052000   A
iv10 = 2*iv5 = -0.064000   A
iv11 = -iv2-iv3+iv4 = -0.242000   A
```

Figure 5: The branch voltages (left) and the branch currents (right) for circuit having dependent sources

References

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[3] MATLAB - User's Guide, MathWorks 2009.

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